Applying fixed-point iteration directly

Note that any equation $f(x)=0$ can be rewritten in many ways as a fixed-point equation $g(x)=x$.

For instance. We can always rewrite $f(x)=0$ as $f(x)+x=x$ (i.e. choose $g(x)=f(x)+x$).

We can then attempt to find a root x^* of $f(x)$ by fixed-point iteration on $g(x)$.

In other words, we start with a value x_0 (an initial approximation) and then compute $x_1, x_2, \ldots$ via $x_{n+1}=g(x_n)$.

Theorem 68 tells us whether that such a fixed-point iteration on $g(x)$ will locally converge to x^*. Moreover, it tells us the order of convergence.

Example 71. Suppose we are interested in computing the roots of $x^2-x-1=0$.

The roots are the golden ratio $\phi=\frac{1}{2}(1+\sqrt{5})\approx 1.618$ and $\psi=\frac{1}{2}(1-\sqrt{5})\approx -0.618$.

There are many ways to rewrite this equation as a fixed-point equation $g(x)=x$. The following are three possibilities:

(a) Rewrite as $x=x^2-1$, so that $g(x)=x^2-1$.

(b) Rewrite first as $x^2=x+1$ and then as $x=1+\frac{1}{x}$, so that $g(x)=1+\frac{1}{x}$.

(c) Rewrite first as $\underbrace{x^2-x}_{=x(x-1)}=1$ and then as $x=\frac{1}{x-1}$, so that $g(x)=\frac{1}{x-1}$.

In each of these three cases and for each root, decide whether fixed-point iteration converges. If it does, determine the order and rate of convergence.

Solution.

(a) In this case, we have $g(x)=x^2-1$ and $g'(x)=2x$.
Since $|g'(\phi)|\approx 3.236>1$ as well as $|g'(\psi)|\approx 1.236>1$, fixed-point iteration does not converge locally to either root.

(b) In this case, we have $g(x)=1+\frac{1}{x}$ and $g'(x)=-\frac{1}{x^2}$.
Since $|g'(\phi)|=\frac{1}{\phi+1}\approx 0.382<1$ and $|g'(\psi)|=\phi+1\approx 2.618>1$, fixed-point iteration converges locally to ϕ but does not converge locally to ψ. Moreover, the convergence to ϕ is linear with rate 0.382.

(c) In this case, we have $g(x)=\frac{1}{x-1}$ and $g'(x)=-\frac{1}{(x-1)^2}$.
Since $|g'(\phi)|=\phi+1\approx 2.618>1$ and $|g'(\psi)|=\frac{1}{\phi+1}\approx 0.382<1$, fixed-point iteration converges locally to ψ but does not converge locally to ϕ. Moreover, the convergence to ψ is linear with rate 0.382.

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Order of convergence of Newton's method

Recall that computing a root x^* of $f(x)$ using Newton's method is equivalent to fixed-point iteration of $g(x) = x - \frac{f(x)}{f'(x)}$.

Comment. In each case, we start with x_0 and iteratively compute $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Theorem 72. Suppose that f is twice continuously differentiable and $f(x^*) = 0$.

- **(typical case)** Newton's method (locally) converges to x^* quadratically with rate $\frac{1}{2}|f''(x^*)/f'(x^*)|$ provided that $f'(x^*) \neq 0$.
- **(troubled case)** If $f'(x^*) = 0$, then Newton's method either does not converge at all or it converges linearly.

Note that, if $f(x^*) = 0$ and $f'(x^*) = 0$, then x^* is a repeated root of $f(x)$. We thus conclude that Newton's method is troubled if we are trying to compute a repeated root.

- **(exceptionally good case)** If $f'(x^*) \neq 0$ and $f''(x^*) = 0$, then Newton's method even converges with order at least 3.

Important comment. In short, Newton's method typically converges quadratically (though in very special cases it can converge even faster) except in the case of repeated roots.

Proof. We apply Theorem 68 to analyze the fixed-point iteration of $g(x) = x - \dfrac{f(x)}{f'(x)}$.

Using the quotient rule we compute that

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

If $f(x^*) = 0$ and $f'(x^*) \neq 0$, then we have $g'(x^*) = 0$. By Theorem 68 this implies that fixed-point iteration converges at least quadratically.

To determine the rate of convergence, we further compute (again using the quotient and product rule) that

$$g''(x) = \frac{(f'(x)f''(x) + f(x)f'''(x))f'(x)^2 - 2f(x)f''(x)f'(x)f''(x)}{f'(x)^4}.$$

From this (unsimplified) expression and $f(x^*) = 0$ we conclude that $g''(x^*) = \dfrac{f''(x^*)}{f'(x^*)}$.

By Theorem 68 this implies that the convergence is quadratic with rate $\dfrac{1}{2}\left|\dfrac{f''(x^*)}{f'(x^*)}\right|$.

Moreover, if $f''(x^*) = 0$ then $g''(x^*) = 0$ so that the convergence is cubic (or higher). □

Example 73. (cont'd) Does Newton's method applied to finding a root of $f(x) = x^3 - 2$ converge locally to $\sqrt[3]{2}$? If so, determine the order and the rate.

This is a continuation of Examples 61 and 70.

Solution. Recall that Newton's method typically converges to x^* with order 2 and rate $\frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right|$.

With $x^* = \sqrt[3]{2}$, we compute $f'(x^*) = 3(x^*)^2 = 3 \cdot 2^{2/3}$. Since $f'(x^*) \neq 0$, we already know that Newton's method converges at least with order 2.

We further compute $f''(x^*) = 6x^* = 6 \cdot 2^{1/3} = 3 \cdot 2^{4/3}$. Since $f''(x^*) \neq 0$, we know that Newton's method does not converge with order larger than 2.

Therefore, Newton's method converges to $x^* = \sqrt[3]{2}$ with order 2 and rate $\frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right| = \frac{1}{2}\frac{3 \cdot 2^{4/3}}{3 \cdot 2^{2/3}} = 2^{-1/3} \approx 0.7937$.

Of course, this matches what we computed in Example 70.

Example 74. $f(x) = e^{-x} - x$ has the unique root $x^* \approx 0.567$. Determine whether Newton's method converges locally to x^*. If it does, what is the order and rate of convergence?

Solution. We compute that $f'(x) = -e^{-x} - 1$ and $f''(x) = e^{-x}$.

Since $x^* = e^{-x^*}$, we have $f'(x^*) = -x^* - 1 \neq 0$.

Hence, by Theorem 72, Newton's method converges to x^* quadratically.

Moreover, the rate is $\frac{1}{2}\left|\frac{f''(x^*)}{f'(x^*)}\right| = \frac{1}{2}\left|\frac{e^{-x^*}}{-e^{-x^*} - 1}\right| = \frac{1}{2}\left|\frac{x^*}{-x^* - 1}\right| \approx 0.181$.

Review. If $f(x^*) = 0$ and $f'(x^*) \neq 0$, then Newton's method (locally) converges to x^* quadratically with rate $\frac{1}{2}|f''(x^*) / f'(x^*)|$.

Note that we can see from here that $f'(x^*) = 0$ is problematic; indeed, in that case, we don't get quadratic convergence (but rather divergence or linear convergence).

We can also see that, if $f''(x^*) = 0$, then we should get even better convergence; indeed, in that case, we get cubic convergence or better.

Example 75. Consider $f(x) = (x - r)(x - 1)(x + 2)$ where r is some constant. Suppose we want to use Newton's method to calculate the root $x^* = 1$.

(a) For which values of r is Newton's method guaranteed to converge (at least) quadratically to $x^* = 1$?

(b) Analyze the cases in which Newton's method does not converge quadratically to $x^* = 1$. Does it still converge? If so, what can we say about the order and rate of convergence?

(c) For which values of r does Newton's method converge to $x^* = 1$ faster than quadratically?

Solution.

(a) We have $f(x) = x^3 - (r - 1)x^2 - (r + 2)x + 2r$ and, hence, $f'(x) = 3x^2 - 2(r - 1)x - (r + 2)$.
Note that $f'(1) = 3 - 3r = 0$ if and only if $r = 1$.
Theorem 72 implies that Newton's method converges (at least) quadratically to $x^* = 1$ if $r \neq 1$.
Comment. Note that $r = 1$ is precisely the case where 1 becomes a double root of $f(x)$.

(b) We need to analyze the case $r = 1$.
In that case $f(x) = (x - 1)^2(x + 2)$ and $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$.
Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{(x - 1)^2(x + 2)}{3(x - 1)(x + 1)} = x - \frac{x^2 + x - 2}{3(x + 1)} = \frac{2}{3}x + \frac{2}{3}\frac{1}{x + 1}.$$

We compute that $g'(x) = \frac{2}{3} - \frac{2}{3}\frac{1}{(x + 1)^2}$ so that, in particular, $g'(1) = \frac{2}{3} - \frac{2}{3}\frac{1}{4} = \frac{1}{2}$.
Since $0 \neq |g'(1)| < 1$ we conclude, by Theorem 68, that Newton's method (locally) converges to $x^* = 1$. Moreover, the convergence is linear with rate $\frac{1}{2}$.
Comment. Since $\frac{1}{2} = 2^{-1}$, this means that we gain roughly one correct binary digit per iteration.

(c) We continue the calculation from the first part. According to Theorem 72, Newton's method converges to 1 faster than quadratic if $f'(1) \neq 0$ and $f''(1) = 0$.
We calculate $f''(x) = 6x - 2(r - 1)$. Thus $f''(1) = 8 - 2r = 0$ if and only if $r = 4$.
Hence, Newton's method converges to 1 faster than quadratic if $r = 4$.

Important comment. Note that what we are observing is exactly as what we should expect: Newton's method typically converges quadratically (though in very special cases it can converge even faster; here, $r = 4$) except in the case of repeated roots (here, $r = 1$).

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