Review. Suppose that x^* is a fixed point of a continuously differentiable function f. The fixed-point iteration

$$x_{n+1} = f(x_n), \quad x_0 = \text{initial approximation},$$

converges to x^* locally if $|f'(x^*)| < 1$.

Example 63. From a plot of $\cos(x)$, we can see that it has a unique fixed point x^* in the interval $[0, 1]$. Does fixed-point iteration converge locally to x^*?

Solution. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. Since $|\sin(x)| < 1$ for all $x \in [0, 1]$, we conclude that $|f'(x^*)| < 1$. By Theorem 60, fixed-point iteration will therefore converge to x^* locally.

Comment. We will continue this analysis in Example 69.

Example 64. Python Let us implement the fixed-point iteration of $\cos(x)$ from the previous example in Python.

```
>>> from math import cos
```

```
>>> def cos_iterate(x, n):
        for i in range(n):
            x = cos(x)
        return x
```

```
>>> [cos_iterate(1, n) for n in range(20)]
```

```
[1, 0.5403023058681398, 0.8575532158463934, 0.6542897904977791, 0.7934803587425656,
0.7013687736227565, 0.7639596829006542, 0.7221024250267077, 0.7504177617637605,
0.7314040424225098, 0.7442373549005569, 0.7356047404363474, 0.7414250866101092,
0.7375068905132428, 0.7401473355678757, 0.7383692041223232, 0.7395672022122561,
0.7387603198742113, 0.7393038923969059, 0.7389377567153445]
```

For comparison. The actual fixed point is $x^* \approx 0.7391$.

Comment. Instead of using a loop, we could also implement the above fixed-point iteration **recursively** in the following way (the recursive part is that the function is calling itself).

```
>>> def cos_iterate_recursively(x, n):
        if n > 0:
            return cos_iterate_recursively(cos(x), n-1)
        return x
```

```
>>> [cos_iterate_recursively(1, n) for n in range(20)]
```

```
[1, 0.5403023058681398, 0.8575532158463934, 0.6542897904977791, 0.7934803587425656,
0.7013687736227565, 0.7639596829006542, 0.7221024250267077, 0.7504177617637605,
0.7314040424225098, 0.7442373549005569, 0.7356047404363474, 0.7414250866101092,
0.7375068905132428, 0.7401473355678757, 0.7383692041223232, 0.7395672022122561,
0.7387603198742113, 0.7393038923969059, 0.7389377567153445]
```

Sometimes recursion results in cleaner code. However the use of loops is usually more efficient.

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Order of convergence

Example 65. Suppose that x_n converges to x^* in such a way that the number of correct digits doubles from one term to the next. What does that mean in terms of the error $e_n = |x_n - x^*|$?

Comment. This is roughly what we observed numerically for the Newton method in Example 47.

Comment. It doesn't matter which base we are using because the number of digits in one base is a fixed constant multiple of the number of digits in another base. Make sure that this clear! (If unsure, how does the number of digits of an integer x in base 2 relate to the number of digits of x in base 10?)

Solution. Recall that the number of correct digits in base b is about $-\log_b(e_n)$.
Doubling these from one term to the next means that $-\log_b(e_{n+1}) \approx -2\log_b(e_n)$.
Equivalently, $\log_b(e_{n+1}) - 2\log_b(e_n) = \log_b\left(\frac{e_{n+1}}{e_n^2}\right) \approx 0$.
This in turn is equivalent to $\frac{e_{n+1}}{e_n^2} \approx 1$.

What if the number of correct digits triples? By the above arguments, we would have $\frac{e_{n+1}}{e_n^3} \approx 1$.
Of course, there is nothing special about 2 or 3.

Example 66. Suppose that x_n converges to x^*. Let $e_n = |x_n - x^*|$ be the error and $d_n = -\log_b(e_n)$ be the number of correct digits (in base b). If $d_{n+1} = A d_n + B$, what does that mean in terms of the error e_n?

Solution. $-\log_b(e_{n+1}) = -A\log_b(e_n) + B$ is equivalent to $\log_b(e_{n+1}) - A\log_b(e_n) = \log_b\left(\frac{e_{n+1}}{e_n^A}\right) = -B$.
This in turn is equivalent to $\frac{e_{n+1}}{e_n^A} = b^{-B}$.

This motivates the following definition.

Definition 67. Suppose that x_n converges to x^*. Let $e_n = |x_n - x^*|$. We say that x_n **converges to x of order q and rate r** if

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^q} = r.$$

Order 1. Convergence of order 1 is called **linear convergence**. As in the previous example, the rate r provides information on the number of additional correct digits per term.

Order 2. Convergence of order 2 is also called **quadratic convergence**. As we saw above, it means that number of correct binary digits d_n roughly doubles from one term to the next. More precisely, $d_{n+1} \approx 2d_n + B$ where the rate $r = 2^{-B}$ tells us that $B = -\log_2(r)$. [Note that r has the advantage of being independent of the base in which we measure the number of correct digits.]

Order of convergence of fixed-point iteration

Theorem 68. Suppose that x^* is a fixed point of a sufficiently differentiable function f. Suppose that $|f'(x^*)| < 1$ so that, by Theorem 60, fixed-point iteration of $f(x)$ converges to x^* locally. Then the convergence is of order M with rate $\frac{1}{M!}|f^{(M)}(x^*)|$ where $M \geqslant 1$ is the smallest integer so that $f^{(M)}(x^*) \neq 0$.

In particular.

- If $f'(x^*) \neq 0$, then the convergence is linear with rate $|f'(x^*)|$.
- If $f'(x^*) = 0$ and $f''(x^*) \neq 0$, then the convergence is quadratic with rate $\frac{1}{2}|f''(x^*)|$.

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Comment. Here, sufficiently differentiable means that f needs to be M times continuously differentiable so that we can apply Taylor's theorem.

Proof. By Taylor's theorem (Theorem 54), if $f'(x^*) = f''(x^*) = \cdots = f^{(M-1)}(x^*) = 0$ for some $M \geqslant 1$, then

$$f(x) = f(x^*) + \frac{1}{M!} f^{(M)}(\xi)(x - x^*)^M$$

for some ξ between x and x^*. We use this with x replaced by x_n to conclude that

$$x_{n+1} - x^* = f(x_n) - f(x^*) = \frac{1}{M!} f^{(M)}(\xi_n)(x_n - x^*)^M$$

for some ξ_n between x_n and x^*.
Thus

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^M} = \frac{1}{M!} f^{(M)}(\xi_n) \quad \underset{n \to \infty}{\longrightarrow} \quad \frac{1}{M!} f^{(M)}(x^*),$$

where the limit follows from the continuity of $f^{(M)}(x)$ (and convergence of $x_n \to x^*$). □

Example 69. (cont'd) From a plot of $\cos(x)$, we can see that it has a unique fixed point in the interval $[0, 1]$. Does fixed-point iteration converge locally? If so, determine the order and the rate.

This is a continuation of Example 63.

Solution. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. Since $|\sin(x)| < 1$ for all $x \in [0, 1]$, we conclude that $|f'(x^*)| < 1$. By Theorem 60, fixed-point iteration will therefore converge to x^* locally.

Since $x^* \approx 0.7391$, we have $|f'(x^*)| \approx |\sin(0.7391)| \approx 0.6736$.

Because $f'(x^*) \neq 0$, we conclude that the order of convergence is 1 and the rate is 0.6736.

Comment. A rate of 0.5 would mean that the number of correct digits increases by 1 for each iteration (and this is what the bisection method provides). Here, convergence is slightly slower.

Example 70. (cont'd)

(a) Newton's method applied to finding a root of $f(x) = x^3 - 2$ is equivalent to fixed-point iteration of which function $g(x)$?

(b) Does Newton's method converge locally to $\sqrt[3]{2}$? If so, determine the order and the rate.

This is a continuation of Example 61. (Also see Example 73 for a an alternative computation for the second part.)

Solution.

(a) Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2}{3x^2} = \frac{2}{3}\left(x + \frac{1}{x^2}\right).$$

(b) By Theorem 60, Newton's method converges locally to $x^* = \sqrt[3]{2}$ if $|g'(x^*)| < 1$.

We compute that $g'(x) = \frac{2}{3} - \frac{4}{3x^3}$ so that $g'(x^*) = \frac{2}{3} - \frac{4}{3 \cdot 2} = 0$.

At this point, we know that Newton's method converges locally to $\sqrt[3]{2}$.

Moreover, $g''(x) = \frac{4}{x^4}$ so that $g''(x^*) = \frac{4}{2^{4/3}} = 2^{2/3} \approx 1.5874$.

Hence, the order of convergence is 2 and the rate is $\frac{1}{2}|g''(x^*)| = \frac{1}{2} \cdot 2^{2/3} = 2^{-1/3} \approx 0.7937$.

Comment. Since the rate is less than 1, the convergence is actually slightly better than a doubling of correct digits for each iteration.

Important. We will see shortly that it is typical for Newton's method to have convergence of order 2.