Review. If $f(x)$ is **analytic** around $x = c$, then it equals its **Taylor series** of $f(x)$ at $x = c$:

$$f(x) \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \ldots$$

Example 57. Consider $f(x) = x e^{-2x}$.

(a) Determine the 2nd Taylor polynomial $p_2(x)$ of $f(x)$ at $x = 0$.

(b) Provide an upper bound for the error $|f(x) - p_2(x)|$ if $x \in \left[0, \frac{1}{10}\right]$.

Solution.

(a) We could determine the 3rd Taylor polynomial by computing derivatives of $x e^{-2x}$.
On the other hand, we already know that $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$ which implies that $x e^{-2x} = x\left(1 - 2x + \frac{2^2}{2}x^2 - \frac{2^3}{6}x^3 + \ldots\right) = x - 2x^2 + 2x^3 - \frac{4}{3}x^4 + \ldots$
We only need the 2nd Taylor polynomial which is $p_2(x) = x - 2x^2$.

(b) If $f(x) = x e^{-2x}$ then Taylor's theorem implies that

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{3!}x^3$$

for some ξ between 0 and x.
We compute that $f^{(3)}(x) = 4(3-2x)e^{-2x}$.
The function $4(3-2x)e^{-2x}$ is decreasing for our values of x (because the derivative is $16(x-2)e^{-2x}$ which is negative for $x \in [0, 2)$) and so the maximum absolute value on $\left[0, \frac{1}{10}\right]$ is taken at $x = 0$ or $x = \frac{1}{10}$.
At $x = 0$, we have $4(3-2x)e^{-2x} = 12$. At $x = \frac{1}{10}$, we have $4(3-2x)e^{-2x} \approx 9.17$.
Therefore, $|f^{(3)}(x)| = |4(3-2x)e^{-2x}| \leqslant 12$.
On the other hand, $|x^3| \leqslant \left(\frac{1}{10}\right)^3$ for all $x \in \left[0, \frac{1}{10}\right]$.
We therefore conclude that the error on $\left[0, \frac{1}{10}\right]$ is bounded by

$$|f(x) - p_2(x)| = \left|\frac{f^{(3)}(\xi)}{3!}x^3\right| \leqslant \frac{12}{3!}\left(\frac{1}{10}\right)^3 = \frac{2}{1{,}000} = 0.002.$$

Fixed-point iteration

Definition 58. x^* is a **fixed point** of a function $f(x)$ if $f(x^*) = x^*$.

Example 59. Determine all fixed points of the function $f(x) = x^3$.

Solution. $x^3 = x$ has the three solutions $x^* = 0, \pm 1$ (and a cubic equation cannot have more than 3 solutions). These are the fixed points.

Idea. Suppose x^* is a fixed point of a continuous function f. If $x_n \approx x^*$, then $f(x_n) \approx f(x^*) = x^* \approx x_n$. If we can guarantee that $f(x_n)$ is closer to x^* than x_n, then we can set

$$x_{n+1} = f(x_n),$$

with the expectation that iterating this process will bring us closer and closer to x^*.

When does this converge? This process converges if $|f(x_n) - x^*| < |x_n - x^*|$ for all x_n close to x^*.

This condition is equivalent to $\left|\frac{f(x_n) - x^*}{x_n - x^*}\right| < 1$.

Since $x^* = f(x^*)$, we have $\frac{f(x_n) - x^*}{x_n - x^*} = \frac{f(x_n) - f(x^*)}{x_n - x^*} \approx f'(x^*)$ provided that x_n is sufficiently close to x^*.

This essentially proves the following result. (See below for a full proof using the mean value theorem.)

Theorem 60. Suppose that x^* is a fixed point of a continuously differentiable function f. If $|f'(x^*)| < 1$, then **fixed-point iteration**

$$x_{n+1} = f(x_n), \quad x_0 = \text{initial approximation},$$

converges to x^* locally.

In that case, we say that x^* is an **attracting fixed point**.

Divergence. If $|f'(x^*)| > 1$, then x^* is a **repelling fixed point**. Our argument shows that fixed-point iteration will not converge to x^* except in the "freak" case where $x_n \not\approx x^*$ but $f(x_n) = x^*$.

Comment. Local convergence means that we have convergence for all initial values x_0 close enough to x^*.

Proof. Note that

$$\begin{aligned} x_{n+1} - x^* &= f(x_n) - f(x^*) \\ &= f'(\xi_n)(x_n - x^*) \end{aligned}$$

where we applied the mean value theorem for the second equation and where ξ_n is between x_n and x^*. Thus

$$|x_{n+1} - x^*| = |f'(\xi_n)| \cdot |x_n - x^*|$$

Since f' is continuous and $|f'(x^*)| < 1$, we have $|f'(x)| < \delta$ for some $\delta < 1$ for all x sufficiently close to x^*. If x_0 is sufficiently to x^* in that sense, then it follows that $|x_1 - x^*| < \delta \cdot |x_0 - x^*|$. In particular, x_1 is even closer to x^* and we can repeat this argument to conclude that $|x_{n+1} - x^*| < \delta \cdot |x_n - x^*|$ for all n. This implies that $|x_n - x^*| < \delta^n \cdot |x_0 - x^*|$. Since $\delta < 1$, this further implies that x_n converges to x^*. □

Newton's method as a fixed-point iteration

Recall that Newton's method for finding a root of $f(x)$ proceeds from an initial approximation x_0 and iteratively computes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Note that this is equivalent to fixed-point iteration of the function $g(x) = x - \frac{f(x)}{f'(x)}$.

Comment. Note that x^* is a fixed point of $g(x) = x - \frac{f(x)}{f'(x)}$ if and only if $\frac{f(x^*)}{f'(x^*)} = 0$.

We have already proven a criterion for convergence of fixed-point iterations (Theorem 60). Our next goal is to develop the tools to analyze the speed of that convergence.

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Example 61. (We will continue this analysis in Example 70.)

(a) Newton's method applied to finding a root of $f(x) = x^3 - 2$ is equivalent to fixed-point iteration of which function $g(x)$?

(b) Determine whether Newton's method converges locally to $\sqrt[3]{2}$.

Solution.

(a) Newton's method applied to $f(x)$ is equivalent to fixed-point iteration of

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2}{3x^2} = \frac{2}{3}\left(x + \frac{1}{x^2}\right).$$

(b) By Theorem 60, Newton's method converges locally to $x^* = \sqrt[3]{2}$ if $|g'(x^*)| < 1$.

Since $g'(x) = \frac{2}{3} - \frac{4}{3x^3}$, we get $g'(x^*) = \frac{2}{3} - \frac{4}{3 \cdot 2} = 0$. Hence Newton's method converges locally to $\sqrt[3]{2}$.

Important comment. Notice that $g'(x^*) = 0$ is, in a way, the strongest sense in which $|g'(x^*)| < 1$. We will see shortly that $g'(x^*) = 0$ implies especially fast convergence of the type we observed in Example 47.

Example 62. (homework)

(a) What are the fixed points of $g(x) = \frac{x}{2} + \frac{1}{x}$?

(b) Does fixed-point iteration of $g(x)$ converge?

(c) Find a function $f(x)$ such that the fixed-point iteration of $g(x)$ is equivalent to Newton's method applied to $f(x)$.

(d) Inspired by the previous parts, suggest a fixed-point iteration to compute square roots.

Solution.

(a) Solving $\frac{x}{2} + \frac{1}{x} = x$, we find $x^2 = 2$ and thus $x = \pm\sqrt{2}$.

Comment. Note that $g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$. Suppose that $x < \sqrt{2}$. Then $2/x > \sqrt{2}$.

When iterating $g(x)$, we are averaging the underestimate and the overestimate, and it is reasonable to expect that the result is a better approximation.

(b) Since $g'(x) = \frac{1}{2} - \frac{1}{x^2}$, we have $g'(\pm\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0$. Hence, both fixed points are attracting fixed points. By Theorem 60, fixed-point iteration of $g(x)$ converges locally to both fixed points.

(c) We are looking for a function $f(x)$ such that $x - \frac{f(x)}{f'(x)} = g(x)$. Equivalently, $\frac{f'(x)}{f(x)} = \frac{1}{x - g(x)} = \frac{2x}{x^2 - 2}$.

This is a first-order differential equation which we can solve for $f(x)$ using separation of variables or by realizing that it is a linear DE. (Our approach below is equivalent to separation of variables.)

Note that $\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x))$. Thus, integrating both sides of the DE,

$$\ln(f(x)) = \int \frac{1}{x - g(x)}\mathrm{d}x = \int \frac{2x}{x^2 - 2}\mathrm{d}x = \ln|x^2 - 2| + C.$$

We conclude that fixed-point iteration of $g(x)$ is equivalent to Newton's method applied to $f(x) = x^2 - 2$.

Comment. The general solution of the DE has one degree of freedom (the C above, which we chose as 0). On the other hand, we know from the beginning that Newton's method applied to $f(x)$ and $Df(x)$ results in the same fixed-point iteration.

(d) Newton's method applied to $f(x) = x^2 - a$ is equivalent to fixed-point iteration of $g(x) = \frac{1}{2}(x + \frac{a}{x})$.

Comment. The resulting method for computing square roots $\sqrt{a}$ is known as the **Babylonian method**. It consists of starting with an approximation $x_0 \approx \sqrt{a}$ and then iteratively computing $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

https://en.wikipedia.org/wiki/Methods_of_computing_square_roots

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