Final: practice

MATH 436/565 — Numerical Analysis
final exam: Monday, Dec 8, 2025

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2. Consider the initial value problem $y' = xy^2 + 1,\ y(1) = 0$.

(a) Approximate the solution $y(x)$ for $x \in [1, 3]$ using Euler's method with 3 steps. In particular, what is the approximation for $y(3)$?

(b) What is the order of the local truncation error? The global error?

(c) Spell out the Taylor method of order 3 for numerically solving this initial value problem.

Solution.

(a) The step size is $h = \frac{3-1}{3} = \frac{2}{3}$. We apply Euler's method with $f(x, y) = xy^2 + 1$:

$$\begin{array}{ll} x_0 = 1 & y_0 = 0 \\ x_1 = \dfrac{5}{3} & y_1 = y_0 + f(x_0, y_0)h = \dfrac{2}{3} \\ x_2 = \dfrac{7}{3} & y_2 = y_1 + f(x_1, y_1)h = \dfrac{2}{3} + \left(\dfrac{5}{3}\left(\dfrac{2}{3}\right)^2 + 1\right)\dfrac{2}{3} = \dfrac{148}{81} \\ x_3 = 3 & y_3 = y_2 + f(x_2, y_2)h = \dfrac{148}{81} + \left(\dfrac{7}{3}\left(\dfrac{148}{81}\right)^2 + 1\right)\dfrac{2}{3} = \dfrac{453914}{59049} \end{array}$$

In particular, the approximation for $y(3)$ is $y_3 = \frac{453914}{59049} \approx 7.68707$.

(b) The local truncation error is of order 2, and the global error is of order 1.

In other words, the local truncation error is $O(h^2)$ and the global error is $O(h)$, where h is the step size.

(c) The Taylor method of order 3 is based on the Taylor expansion

$$y(x + h) = y(x) + y'(x)h + \frac{1}{2}y''(x)h^2 + \frac{1}{6}y'''(x)h^3 + O(h^4),$$

where we have a local truncation error of $O(h^4)$ so that the global error will be $O(h^3)$.

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From the DE we know that $y'(x) = xy^2 + 1$. We differentiate this to obtain

$$y''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(xy^2 + 1) = y^2 + 2xyy' = y^2 + 2xy \cdot (xy^2 + 1) = y^2 + 2x^2y^3 + 2xy .$$

We differentiate once more to find

$$\begin{aligned} y'''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(y^2 + 2x^2y^3 + 2xy) = 2yy' + 4xy^3 + 6x^2y^2y' + 2y + 2xy' \\ &= (2y + 6x^2y^2 + 2x)(xy^2 + 1) + 4xy^3 + 2y \\ &= 2(x + 2y + 4x^2y^2 + 3xy^3 + 3x^3y^4) \end{aligned}$$

(in the second step we replaced y' by $xy^2 + 1$).

Hence, the Taylor method of order 3 takes the form:

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + (x_ny_n^2 + 1)h + \tfrac{1}{2}(y_n^2 + 2x_n^2y_n^3 + 2x_ny_n)h^2 + \tfrac{1}{3}(x_n + 2y_n + 4x_n^2y_n^2 + 3x_ny_n^3 + 3x_n^3y_n^4)h^3 \end{aligned}$$

with the initial values $x_0 = 1$ and $y_0 = 0$.

Problem 3. Using a step size of $h = \frac{1}{3}$, discretize the Dirichlet problem $\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= 3 \\ u(x,1) &= 5 \\ u(0,y) &= 1 \\ u(1,y) &= 2 \end{aligned}$ where $x \in (0,1)$ and $y \in (0,1)$.

Spell out a system of linear equations for the resulting lattice points. Do not solve that system.

(Note that, for the Dirichlet problem as well as for our discretization, it doesn't matter that the boundary conditions aren't well-defined at the corners.)

Solution. Note that our region is a square with side lengths 1 (in both x and y directions). We write $u_{m,n} = u(mh, nh)$. Make a sketch!

$$\begin{array}{cccc} & 5 & 5 & \\ 1 & u_{1,2} & u_{2,2} & 2 \\ 1 & u_{1,1} & u_{2,1} & 2 \\ & 3 & 3 & \end{array}$$

We need to determine equations for the four unknowns $u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}$.

If we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx} + u_{yy} = 0$ translates into

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = 0.$$

We get one such equation for each of the four unknowns (where the unknown is multiplied by -4). For instance, the equation for $u_{2,1}$ is

$$u_{1,1} + \underbrace{u_{3,1}}_{=2} + \underbrace{u_{2,0}}_{=3} + u_{2,2} - 4u_{2,1} = 0.$$

Spelling out these equation in matrix-vector form (the above equation for $u_{2,1}$ is the second row), we obtain:

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -6 \\ -7 \end{bmatrix}$$

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Problem 4. Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on the polygonal region with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$, $(2,2)$, $(0,2)$. Suppose that $u(0,y) = 5$ for $y \in (0,2)$ and that $u(x,y) = 7$ for all other points (x,y) on the boundary of the region. Discretize this Dirichlet problem using a step size of $h = \frac{1}{2}$.

Spell out a system of linear equations for the resulting lattice points. Do not solve that system.

Solution. We write $u_{m,n} = u(mh, nh)$. Make a sketch!

$$\begin{array}{ccccc} & 7 & 7 & 7 & \\ 5 & u_{1,3} & u_{2,3} & u_{3,3} & 7 \\ 5 & u_{1,2} & 7 & 7 & \\ 5 & u_{1,1} & 7 & & \\ & 7 & & & \end{array}$$

If we approximate $u_{xx} + u_{yy}$ by $\frac{1}{h^2}[u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx} + u_{yy} = 0$ translates into

$$u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = 0.$$

Spelling out these equation in matrix-vector form, we obtain:

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} -19 \\ -12 \\ -12 \\ -14 \\ -21 \end{bmatrix}$$

Comment. Note that, because of the way we discretize, it matters that there is a well-defined temperature at the boundary vertex $(1,1)$. For the other vertices, we don't need a well-defined temperature (and so it is not a problem that it is unclear what the temperature should be at $(0,0)$ or $(0,2)$ where it jumps from 5 to 7).

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