Final: practice

MATH 311 — Intro to Number Theory
final exam: Thursday, Nov 19

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2.

(a) Which real numbers have a finite continued fraction?

(b) Which number is represented by the continued fraction $[1;2,1,2,1,2]$?

(c) Determine all convergents of $[1;2,1,2,1,2]$.

(d) Which number is represented by the infinite continued fraction $[1;2,1,2,1,2,1,2,\ldots]$?

(e) Compare, numerically, the first six convergents (computed above) to the value of the infinite continued fraction.

Solution.

(a) These are precisely the rational numbers.

(b) $[1;2,1,2,1,2] = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{2}}}}} = \frac{41}{30}$

Comment. Of course, we can simplify this continued fraction directly. But that is a bit time consuming and prone to errors. A better is way is to compute the convergents recursively as we do in the next part.

(c) The convergents are $C_0 = 1$, $C_1 = [1;2] = 1 + \frac{1}{2} = \frac{3}{2}$, $C_2 = [1;2,1] = 1 + \frac{1}{2 + \frac{1}{1}} = \frac{4}{3}$.

We can continue like that but the computations will get more involved. Instead, we should proceed recursively. Recall from class that the convergents $C_n = \frac{p_n}{q_n}$ of $[a_0; a_1, a_2, \ldots]$ are characterized by

$$\begin{array}{lcl} p_k = a_k p_{k-1} + p_{k-2} & \quad\text{and}\quad & q_k = a_k q_{k-1} + q_{k-2} \\ \text{with } p_{-2} = 0, \quad p_{-1} = 1 & & \text{with } q_{-2} = 1, \quad q_{-1} = 0 \end{array}.$$

Armin Straub
straub@southalabama.edu

The corresponding calculations of p_n and q_n are as follows:

n	-2	-1	0	1	2	3	4	5
a_n			1	2	1	2	1	2
p_n	0	1	1	3	4	11	15	41
q_n	1	0	1	2	3	8	11	30
C_n			1	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{11}{8}$	$\frac{15}{11}$	$\frac{41}{30}$

(d) Write $x = [1; 2, 1, 2, 1, 2, 1, 2, \ldots]$. Then, $x = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \ldots}}}} = 1 + \frac{1}{2 + \frac{1}{x}}$.

The equation $x = 1 + \frac{1}{2 + \frac{1}{x}}$ simplifies to $x - 1 = \frac{x}{2x + 1}$. Further (note that, clearly $x \neq -\frac{1}{2}$ so that $2x + 1 \neq 0$) simplifies to $(x - 1)(2x + 1) = x$ or $2x^2 - 2x - 1 = 0$, which has the solutions $x = \frac{2 \pm \sqrt{4 + 8}}{4} = \frac{1 \pm \sqrt{3}}{2}$.

Since $\frac{1 + \sqrt{3}}{2} \approx 1.366$ and $\frac{1 - \sqrt{3}}{2} \approx -0.366$, we conclude that $[1; 2, 1, 2, 1, 2, 1, 2, \ldots] = \frac{1 + \sqrt{3}}{2}$.

(e) $C_0 = 1$, $C_1 = \frac{3}{2} = 1.5$, $C_2 = \frac{4}{3} \approx 1.333$, $C_3 = \frac{11}{8} = 1.375$, $C_4 = \frac{15}{11} \approx 1.364$, $C_5 = \frac{41}{30} \approx 1.367$

These values quickly approach $\frac{1 + \sqrt{3}}{2} \approx 1.366$ in the expected alternating fashion.

Problem 3.

(a) Express the numbers $\frac{252}{193}$ and $-\frac{337}{221}$ as a simple continued fraction.

(b) Is this the unique simple continued fraction representing $\frac{252}{193}$? Explain!

Solution.

(a) The simplest way to obtain the continued fraction for $\frac{252}{193}$ is via the Euclidean algorithm:

$$252 = \boxed{1} \cdot 193 + 59, \quad 193 = \boxed{3} \cdot 59 + 16, \quad 59 = \boxed{3} \cdot 16 + 11, \quad 16 = \boxed{1} \cdot 11 + 5, \quad 11 = \boxed{2} \cdot 5 + 1, \quad 5 = \boxed{5} \cdot 1 + 0$$

Hence, $\frac{252}{193} = [1; 3, 3, 1, 2, 5]$.

To determine a simple continued fraction for $-\frac{337}{221}$, we first write $-\frac{337}{221} = -2 + \frac{105}{221} = \boxed{-2} + \frac{1}{\frac{221}{105}}$. We then proceed using the Euclidean algorithm applied to $\frac{221}{105}$.

$$221 = \boxed{2} \cdot 105 + 11, \quad 105 = \boxed{9} \cdot 11 + 6, \quad 11 = \boxed{1} \cdot 6 + 5, \quad 6 = \boxed{1} \cdot 5 + 1, \quad 5 = \boxed{5} \cdot 1 + 0.$$

Combined, $-\frac{337}{221} = [-2; 2, 9, 1, 1, 5]$.

(b) No, a finite continued fraction can always be expressed in two ways because of the simple relation $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$, assuming $a_n > 1$. In this case, we also have $\frac{252}{193} = [1; 3, 3, 1, 2, 4, 1]$.

Armin Straub
straub@southalabama.edu