Example 1. **($3x+1$ problem)** Consider the function

$$T(x)=\begin{cases} 3x+1, & \text{if } x \text{ is odd,} \\ x/2, & \text{if } x \text{ is even.} \end{cases}$$

Fix a value N, and define inductively $a_1=N$, $a_{n+1}=T(a_n)$. That is, the sequence $a_1, a_2, a_3, \ldots$ equals $N, T(N), T(T(N)), T(T(T(N)))\ldots$.

Experimenting.

- For $N=1$, we get $1, 4, 2, 1, 4, 2, 1, \ldots$ (repeating the values $1, 4, 2$)
- For $N=2$, we get $2, 1, \ldots$ (and then repeating as for $N=1$)
- For $N=3$, we get $3, 10, 5, 16, 8, 4, 2, 1, \ldots$
- For $N=4$, we get $4, 2, 1, \ldots$
- For $N=5$, we get $5, 16, 8, 4, 2, 1, \ldots$ (we had already seen that for $N=3$)
- For $N=6$, we get $6, 3, \ldots$ (and we know what happens after 3)
  Likewise, any even starting point will not produce anything new.
- For $N=7$, we get $7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \ldots$ (16 steps total)

Conjecture. For any starting value N, the sequence a_n eventually cycles $\ldots, 1, 4, 2, \ldots$.

How many steps does it take to reach 1? From our experiments above, the numbers of steps it takes for small N (from 1 to 7) are: $0, 1, 7, 2, 5, 8, 16, \ldots$ For $N=97$, it takes a whopping 118 steps to get to 1.

History. Conjectured by Collatz 1937 and Thwaites in the 50's. The latter offers 1000 pound for a solution.

H.S.M. Coxeter offers 50 USD for a proof and 100 USD for a counterexample.

Over 100 publications related to the problem. Famous mathematicians warning about its difficulty: "Mathematics is not yet ready for such problems,", "Hopeless. Absolutely hopeless.", (Paul Erdös), "Don't try to solve these problems!" (Richard Guy)

If wrong. There are two possibilities, the Collatz conjecture could be wrong: a sequence could grow without bounds, or it could run into a different cycle. It is not known whether one of these is impossible. It is even possible that there could be infinitely many cycles besides $1, 4, 2$.

Evidence. The conjecture is true for all $N < 87 \cdot 2^{60} \approx 10^{20}$ (status as of 2019).

> If your problem generates integers, it is always worth a try to look up those integers in the **On-Line Encyclopedia of Integer Sequences**.
>
> https://oeis.org/
>
> You might find references, formulas, related problems that result in the same integers, ...

Example 2. Suppose you had never heard about the $3x+1$ problem as being a thing. How could you find out about it on the OEIS?

Solution. There often is many possibilities. For instance, we could put in:

- $7, 22, 11, 34, 17, 52, 26$ (the beginning of the values starting at 7)
  And, indeed, there are 12 matching entries in the OEIS (as of 8/2019), most of which reference the Collatz problem.
- $0, 1, 7, 2, 5, 8, 16$ (the number of steps to reach 1)
  This results in 2 matches, both of which concern the Collatz problem.

Comment. The OEIS typically focuses on integer sequences (infinite lists of integers), so the second option was a bit more promising.

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One of our topics will be primes and factoring, particularly finding common factors. From our familiarity with integers, the following theorem probably feels rather obvious to you.

Theorem 3. (Fundamental Theorem of Arithmetic) Every integer $n > 1$ can be written as a product of primes. This factorization is unique (apart from the order of the factors).

For instance. $21 = 3 \cdot 7$ or $100 = 2^2 \cdot 5^2$

The following (advanced, and just for fun and perspective!) example is meant to illustrate that the idea of factorization into primes and the uniqueness of such factorizations should not be taken entirely for granted.

Example 4.

- In more advanced number theory, it is common to extend the set of integers. For instance, the **Gaussian integers** are numbers of the form $a + bi$, where a and b are ordinary integers and i is the imaginary unit satisfying $i^2 = -1$.

  Note that 5 is no longer a prime because we have $5 = (2 + i)(2 - i)$. It turns out that the quantities $2 \pm i$ cannot be further factored. They are primes in this setting.

  [These claims are usually proved by introducing the "norm" $N(a + bi) = a^2 + b^2$. This function is multiplicative, meaning that $N(xy) = N(x)N(y)$. It follows that $2 + i$ must be a prime because $N(2 + i) = 5$ is a prime. For contrast, $N(5) = 25$ is not a prime.]

  https://en.wikipedia.org/wiki/Table_of_Gaussian_integer_factorizations

- A similar kind of integers consists of numbers of the form $a + bi\sqrt{5}$, where a and b are ordinary integers.

  Note that $i\sqrt{5}$ solves the equation $x^2 + 5 = 0$, and so is very similar to i which solves $x^2 + 1 = 0$.

  [The numbers $a + bi\sqrt{5}$ are called the ring of integers of the field $\mathbb{Q}(\sqrt{-5})$.]

  Then we have two different factorizations of 6, namely,

  $$6 = 2 \cdot 3, \quad 6 = \left(1 + i\sqrt{5}\right)\left(1 - i\sqrt{5}\right).$$

  The numbers 2, 3, $1 \pm i\sqrt{5}$ cannot be factored further.

  [They are called irreducible. However, technically speaking, they are not primes. There is a subtle distinction between these two concepts that is not visible when working with ordinary integers.]

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