18 Continued fractions

Definition 138. A **continued fraction** is a fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}},$$

with $a_1, a_2, \ldots$ positive. Written as $[a_0; a_1, a_2, \ldots]$.
Called **simple** if all the a_i are integers.

Example 139. Express $\frac{5}{3}$ as a simple continued fraction.

Solution. $\frac{5}{3} = 1 + \frac{2}{3} = 1 + \frac{1}{3/2} = 1 + \frac{1}{1 + \frac{1}{2}} = [1; 1, 2]$

Writing the final 2 as $1 + \frac{1}{1}$, we also have $\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = [1; 1, 1, 1]$.

More generally. If $a_n > 1$, we always have $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$.

Comment. Apart from these two variations, the simple continued fraction for $\frac{5}{3}$ is unique.

Note that we are used to a similar ambiguity when dealing with terminating decimal expansions: for instance, $1.25000000\ldots = 1.24999999\ldots$

A slight variation. It follows from the above that $\frac{3}{5} = 0 + \frac{1}{5/3} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = [0; 1, 1, 2]$.

More generally, we always have that, if $x = [a_0; a_1, a_2, \ldots]$ with $a_0 > 0$, then $\frac{1}{x} = [0; a_0, a_1, a_2, \ldots]$.

Example 140. Express $\frac{43}{19}$ as a simple continued fraction.

Solution. $\frac{43}{19} = 2 + \frac{5}{19} = 2 + \frac{1}{19/5} = 2 + \frac{1}{3 + \frac{4}{5}} = 2 + \frac{1}{3 + \frac{1}{5/4}} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}} = [2; 3, 1, 4]$

Again, also, $\frac{43}{19} = [2; 3, 1, 4] = \left[2; 3, 1, 3 + \frac{1}{1}\right] = [2; 3, 1, 3, 1]$.

Super important comment. We have done this computation before (in a different guise)!

By the Euclidean algorithm: $43 = \boxed{2} \cdot 19 + 5$, $19 = \boxed{3} \cdot 5 + 4$, $5 = \boxed{1} \cdot 4 + 1$, $4 = \boxed{4} \cdot 1 + 0$.

Example 141. Evaluate $[2; 3]$, $[2; 3, 4]$, and $[2; 3, 4, 5]$.

Solution.

$[2; 3] = 2 + \frac{1}{3} = \frac{7}{3} \approx 2.333$

$[2; 3, 4] = 2 + \frac{1}{3 + \frac{1}{4}} = 2 + \frac{4}{13} = \frac{30}{13} \approx 2.308$

$[2; 3, 4, 5] = 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}} = 2 + \frac{1}{3 + \frac{5}{21}} = 2 + \frac{21}{68} = \frac{157}{68} \approx 2.309$

Definition 142. The **convergents** C_k of $[a_0; a_1, a_2, \ldots]$ are the truncated continued fractions $C_0 = a_0$, $C_1 = [a_0; a_1]$, $C_2 = [a_0; a_1, a_2]$, $C_k = [a_0; a_1, a_2, a_3]$, …

Armin Straub
straub@southalabama.edu

Theorem 143. The convergents C_k of a simple continued fraction $[a_0;\ a_1,\ a_2,\ \ldots]$ always converge to a value x in the following alternating fashion

$$C_0 < C_2 < C_4 < \cdots \quad x \quad \cdots < C_5 < C_3 < C_1.$$

We simply write $x = [a_0; a_1, a_2, \ldots]$ for that value.

If the continued fraction is finite, that is $x = [a_0; a_1, a_2, \ldots, a_n]$, then we only have the convergents $C_0, C_1, \ldots, C_n$ and $C_n = x$.

Proof. From

$$C_0 = a_0, \quad x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}, \quad C_1 = a_0 + \frac{1}{a_1},$$

we see that C_0 is less than x, as well as less than all other convergents (because all of these equal a_0 plus something positive). Similarly, C_1 is larger than x, as well as larger than all other convergents.

The full claim then follows by, likewise, looking at $[a_1; a_2, a_3 \ldots]$ in place of $[a_0; a_1, a_2, \ldots]$.

See Theorem 15.4 in our book for full details. □

Theorem 144. (representing a real number as a simple continued fraction)

- An irrational number x has a unique representation as a simple continued fraction. This continued fraction is infinite.
- A rational number x has exactly two representations as a simple continued fraction. Both are finite (one ends in a 1 and the other doesn't).

Proof. Let x be a positive real number. Let us think about how a continued fraction for x has to look like. [The argument for negative x is essentially the same. For negative x, a_0 will be negative but the remainder and the other digits are positive.]

As in Theorem 143, we have $C_0 \leqslant x \leqslant C_1$ where $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1} \leqslant a_0 + 1$.

Hence, $a_0 \leqslant x \leqslant a_0 + 1$ which means that a_0 has to be the integer $a_0 = \lfloor x \rfloor$.

(unless) Well, unless x is an integer itself, in which case we have the two possibilities $a_0 = x$ or $a_0 = x - 1$. But in that special case, we are done: the continued fraction for x is finite and there is exactly the two representations $x = [x]$ and $x = [x - 1; 1]$.

So, $x = a_0 + \frac{1}{y}$ with $y = \frac{1}{x - a_0} > 0$, and the continued fraction for x is $x = [a_0; b_0, b_1, \ldots]$ provided that y has the continued fraction $y = [b_0; b_1, \ldots]$. We now repeat our argument, starting with the positive real number y (so that $b_0 = \lfloor y \rfloor$, ...).

There is two possibilities:

- The process stops along the way because the number we are looking at is an integer (the "unless" case). In that case, we get exactly two finite simple continued fractions for x (one of which ends in 1).

  This happens if and only if x is rational (from the Euclidean algorithm we know that every rational number has a finite simple continued fraction; conversely, a finite simple continued fraction necessarily represents a rational number).
- The process continues indefinitely. In that case, we get a (unique) infinite simple continued fraction for x. (By Theorem 143, this continued fraction indeed converges to x.) □

Review. Euler's number $e = 2.71828182846\ldots$ and its significance (differential equations, compound interest)