Midterm #1

MATH 311 — Intro to Number Theory
Tuesday, Sep 20

Please print your name:

No notes or tools of any kind are permitted. There are 24 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (warmup, 2 points) Write down the first 5 rows of the Pascal triangle. Use this to expand $(x+y)^5$.

Solution.

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

□

Problem 2. (warmup, 2 points) The sequence $A(n)$ is defined by $A(1)=1$, $A(2)=1$ and

$$A(n) = n \cdot A(n-1) - A(n-2)$$

for $n > 2$. Determine $A(4)$.

(This sequence appears, for instance, in the context of Bessel functions [D.H. Lehmer, *Annals of Mathematics* 33(1), 1932].)

Solution.

$$\begin{aligned} A(3) &= 3 \cdot A(2) - A(1) = 3 - 1 = 2 \\ A(4) &= 4 \cdot A(3) - A(2) = 8 - 1 = 7 \end{aligned}$$

□

Problem 3. (1+3 points)

(a) Gauss' summation formula states that $\sum_{k=1}^{n} k = \boxed{}$.

(b) Derive a formula for the sum $\sum_{k=0}^{n} (3k+2)$ from Gauss' summation formula.

Solution.

(a) $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

(b) $\sum_{k=0}^{n} (3k+2) = \sum_{k=0}^{n} (3k) + \sum_{k=0}^{n} 2 = 3\sum_{k=0}^{n} k + 2(n+1) \overset{\text{(Gauss)}}{=} 3\frac{n(n+1)}{2} + 2(n+1) = \frac{n+1}{2}(3n+4)$

(Alternatively, we could proceed from scratch like little Gauss. See solutions to first homework set.) □

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Problem 4. (1 point) Which positive integers n have the following property: for all integers a, b,

$$\text{if } n|ab, \quad \text{then } n|a \text{ or } n|b.$$

Solution. A positive integers n has this property if and only if n is 1 or a prime. □

Problem 5. (3 points) Which are the possible remainders that the square of an integer leaves upon division by 4?

Solution. Since we are dividing by 4, it is natural to distinguish the following four cases:

- If $x = 4q$, then $x^2 = 16q^2$ leaves remainder 0.
- If $x = 4q + 1$, then $x^2 = 16q^2 + 8q + 1$ leaves remainder 1.
- If $x = 4q + 2$, then $x^2 = 16q^2 + 16q + 4$ leaves remainder 0.
- If $x = 4q + 3$, then $x^2 = 16q^2 + 24q + 9$ leaves remainder 1.

In summary, the only possible remainders are $0, 1$. (Remainders $2, 3$ are not possible.) □

Problem 6. (4+1 points)

(a) Find $d = \gcd(32, 50)$. Using the Euclidean algorithm, find integers x, y such that $32x + 50y = d$.

(b) For which values of k has the diophantine equation $32x + 50y = k$ at least one integer solution?

Solution.

(a) $d = \underbrace{\gcd(32, 50)}_{50=1\cdot 32+18} = \underbrace{\gcd(18, 32)}_{32=1\cdot 18+14} = \underbrace{\gcd(14, 18)}_{18=1\cdot 14+4} = \underbrace{\gcd(4, 14)}_{14=3\cdot 4+2} = \gcd(2, 4) = 2$

We trace back through the algorithm to find

$$2 = 14 - 3 \cdot \underbrace{4}_{4=18-1\cdot 14} = 4 \cdot \underbrace{14}_{14=32-1\cdot 18} - 3 \cdot 18 = 4 \cdot 32 - 7 \cdot \underbrace{18}_{18=50-1\cdot 32} = 11 \cdot 32 - 7 \cdot 50.$$

Hence, $x = 11$ and $y = -7$ satisfy $32x + 50y = d$.

(b) Since $\gcd(32, 50) = 2$, the diophantine equation $32x + 50y = k$ has solutions if and only if $2|k$. □

Problem 7. (3 points) Determine all solutions of $8x + 14y = 4$ with x and y integers.

Solution. We first simplify the equation to $4x + 7y = 2$, so that $\gcd(4, 7) = 1$.

We see that $x = 2$, $y = -1$ is a solution to $4x + 7y = 1$ (you can, of course, use the Euclidean algorithm if you wish).

Hence, a particular solution to $4x + 7y = 2$ is given by $x = 4$, $y = -2$.

The general solution to $4x + 7y = 2$ is thus given by $x = 4 + 7t$, $y = -2 - 4t$, where t can be any integer. □

Problem 8. (4 points) Using induction, prove that $15|2^{4n} - 1$ for all integers $n \geqslant 0$.

Carefully state all steps of inductions to receive full credit!

Solution. We use induction on the claim that $15|(2^{4n} - 1)$.

- The base case $(n = 0)$ is that $15|(1 - 1)$. That's true.
- For the inductive step, assume that $15|(2^{4n} - 1)$ is true for some value of n.

  We need to show that $15|(2^{4(n+1)} - 1)$ as well. Indeed,

$$\begin{aligned} 2^{4(n+1)} - 1 &= 2^4 \cdot 2^{4n} - 1 \\ &= 2^4(2^{4n} - 1) + 2^4 - 1 \\ &= 2^4(2^{4n} - 1) + 15 \end{aligned}$$

  is divisible by 15 because:

  - $15|15$ for obvious reasons, and
  - $15|2^4(2^{4n} - 1)$ by the induction hypothesis.

By induction, it follows that $15|(2^{4n} - 1)$ for all integers $n \geqslant 0$. □

(extra scratch paper)

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