Midterm #1: practice

MATH 311 — Intro to Number Theory
midterm: Tuesday, Sep 20

Please print your name:

Problem 1. Using induction, prove that $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for all integers $n \geqslant 0$.

Solution. Write $s(n) = 1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2$.
We use induction on the claim that $s(n) = \frac{(n+1)(2n+1)(2n+3)}{3}$.

- The base case $(n = 0)$ is that $s(0) = 1^2 = \frac{1 \cdot 1 \cdot 3}{3}$. That's true.
- For the inductive step, assume the formula holds for some value of n.

  We need to show the formula also holds for $n+1$.

$$\begin{aligned} s(n+1) &= s(n) + (2(n+1)+1)^2 \\ \text{(using the induction hypothesis)} &= \frac{(n+1)(2n+1)(2n+3)}{3} + (2n+3)^2 \\ &= \frac{2n+3}{3}[(n+1)(2n+1) + 3(2n+3)] \\ &= \frac{2n+3}{3}[2n^2 + 9n + 10] = \frac{2n+3}{3}(n+2)(2n+5) \\ &= \frac{(n+2)(2n+3)(2n+5)}{3} \end{aligned}$$

  This shows that the formula also holds for $n+1$.

By induction, the formula is true for all integers $n \geqslant 0$. □

Problem 2. Which are the possible remainders that the square of an integer leaves upon division by 6?

Solution. Since we are dividing by 6, it is natural to distinguish the following six cases:

- If $x = 6q$, then $x^2 = 36q^2$ leaves remainder 0.
- If $x = 6q + 1$, then $x^2 = 36q^2 + 12q + 1$ leaves remainder 1.
- If $x = 6q + 2$, then $x^2 = 36q^2 + 24q + 4$ leaves remainder 4.
- If $x = 6q + 3$, then $x^2 = 36q^2 + 36q + 9$ leaves remainder 3.
- If $x = 6q + 4$, then $x^2 = 36q^2 + 48q + 16$ leaves remainder 4.
- If $x = 6q + 5$, then $x^2 = 36q^2 + 60q + 25$ leaves remainder 1.

In summary, the only possible remainders are $0, 1, 3, 4$. (Remainders $2, 5$ are not possible.) □

Problem 3. Find $d = \gcd(119, 272)$. Using the Euclidean algorithm, find integers x, y such that $119x + 272y = d$.

Solution. The gcd is

$$d = \underbrace{\gcd(119, 272)}_{272 = 2 \cdot 119 + 34} = \underbrace{\gcd(34, 119)}_{119 = 3 \cdot 34 + 17} = \gcd(17, 34) = 17.$$

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We trace back through the Euclidean algorithm to find integers x, y such that $119x+272y=17$:

$$17=\underbrace{119-3\cdot 34}_{34=272-2\cdot 119}=7\cdot 119-3\cdot 272$$

So, here, $x=7$ and $y=-3$.

Comment. Note the other values also work for x and y. In fact, the general solution is $x=7+\frac{272}{17}t$, $y=-3-\frac{119}{17}t$. $\square$

Problem 4.

(a) For which values of k has the diophantine equation $123x+360y=k$ at least one integer solution?

(b) Determine all solutions of $123x+360y=99$ with x and y positive integers.

Solution.

(a) We first compute $\gcd(123, 360)$ and find

$$\underbrace{\gcd(123,360)}_{360=2\cdot 123+114}=\underbrace{\gcd(114,123)}_{123=1\cdot 114+9}=\underbrace{\gcd(9,114)}_{114=12\cdot 9+6}=\underbrace{\gcd(6,9)}_{9=1\cdot 6+3}=\gcd(3,6)=3.$$

We therefore see that the diophantine equation $123x+360y=k$ has at least one integer solution if and only if k is a multiple of 3.

(b) Since $3|99$, the diophantine equation $123x+360y=99$ has solutions. We first divide out the common factor of 3 to get the simplified equation $41x+120y=33$.

We already know that $\gcd(41, 120)=1$ but to find integers x, y such that $41x+120y=1$, we go through the Euclidean algorithm again (if you want, you could reuse our previous computation; note that everything is the same just with the common factor of 3 cancelled everywhere):

$$\underbrace{\gcd(41,120)}_{120=2\cdot 41+38}=\underbrace{\gcd(38,41)}_{41=1\cdot 38+3}=\underbrace{\gcd(3,38)}_{38=12\cdot 3+2}=\underbrace{\gcd(2,3)}_{3=1\cdot 2+1}=\gcd(1,2)=1.$$

We trace back through the algorithm to find

$$1=\underbrace{3-1\cdot 2}_{2=38-12\cdot 3}=\underbrace{-1\cdot 38+13\cdot 3}_{3=41-1\cdot 38}=\underbrace{13\cdot 41-14\cdot 38}_{38=120-2\cdot 41}=-14\cdot 120+41\cdot 41.$$

In other words, $41x+120y=1$ has the solution $x=41$, $y=-14$.

Multiplying this equation with 33, we find that our original equation $41x+120y=33$ has the particular solution $x=33\cdot 41$, $y=-33\cdot 14$.

The general solution is $x=33\cdot 41+120t$, $y=-33\cdot 14-41t$ with t any integer.

However, we are only interested in solutions with $x>0$ and $y>0$. $x>0$ means $t>-\frac{33\cdot 41}{120}=-12+\frac{29}{40}$ (that is, $t\in\{-11,-10,-9,...\}$), while $y>0$ means $t<-\frac{33\cdot 14}{41}=-12+\frac{30}{41}$ (that is, $t\in\{-12,-13,-14,...\}$). These conditions contradict each other, which means that there are no solutions with both x and y positive integers. $\square$

Problem 5. We call (a, b, c) a prime triple if a, b, c are all primes.

(a) List the first few prime triples of the form $(p, p+2, p+6)$.
(It is believed, but nobody can show, that there are infinitely many such triples.)

(b) Show that there is only a single prime triple of the form $(p, p+2, p+4)$.

(c) Show that there are no prime triples of the form $(p, p+2, p+5)$.

Solution.

(a) $(5,7,11)$, $(11,13,17)$, $(17,19,23)$, $(41,43,47)$, $(101,103,107)$, ...

(b) You showed in Homework 2 that, for any integer x, one of the integers x, $x+2$, $x+4$ is divisible by 3. Hence, in each triple $(p,p+2,p+4)$, one of the three numbers is divisible by 3.

This leaves $p=3$ as the only possibility and, indeed, $(3,5,7)$ is a prime triple.

(c) Either p or $p+5$ is an even number, and hence divisible by 2.

This leaves $p=2$ as the only possibility, but $(2,4,7)$ is not a prime triple. □

Problem 6. Using induction, prove that $5|(3^{3n+1}+2^{n+1})$ for all integers $n\geqslant 0$.

Solution. We use induction on the claim that $5|(3^{3n+1}+2^{n+1})$.

- The base case $(n=0)$ is that $5|(3^1+2^1)$. That's true.
- For the inductive step, assume that $5|(3^{3n+1}+2^{n+1})$ is true for some value of n.

  We need to show that $5|(3^{3n+4}+2^{n+2})$ as well. Indeed,

$$\begin{aligned} 3^{3n+4}+2^{n+2} &= 27\cdot 3^{3n+1}+2\cdot 2^{n+1} \\ &= 27\cdot 3^{3n+1}+2(3^{3n+1}+2^{n+1}-3^{3n+1}) \\ &= 25\cdot 3^{3n+1}+2(3^{3n+1}+2^{n+1}) \end{aligned}$$

  is divisible by 5 because:

  - $5|25\cdot 3^{3n+1}$ for obvious reasons, and
  - $5|(3^{3n+1}+2^{n+1})$ by the induction hypothesis.

By induction, it follows that $5|(3^{3n+1}+2^{n+1})$ for all integers $n\geqslant 0$. □

Problem 7. Determine, and prove, a formula for the sum $\sum_{k=0}^{n}(4k+1)$.

Solution. $\sum_{k=0}^{n}(4k+1)=\sum_{k=0}^{n}(4k)+\sum_{k=0}^{n}1=4\sum_{k=0}^{n}k+(n+1)\overset{\text{(Gauss)}}{=}4\frac{n(n+1)}{2}+(n+1)=(n+1)(2n+1)$

(Alternatively, we could proceed from scratch like little Gauss. See solutions to first homework set.) □

Problem 8. Using induction, show that any two consecutive Fibonacci numbers are relatively prime. In other words, show that $\gcd(F_n,F_{n+1})=1$ for all integers $n\geqslant 1$.

Solution. We use induction on the claim that $\gcd(F_n,F_{n+1})=1$.

- The base case $(n=1)$ is that $\gcd(F_1,F_2)=1$. That's true because $F_1=F_2=1$.
- For the inductive step, assume that $\gcd(F_n,F_{n+1})=1$ is true for some value of n.

  We need to show that $\gcd(F_{n+1},F_{n+2})=1$ as well. Indeed:

$$\begin{aligned} \gcd(F_{n+1},F_{n+2}) &= \gcd(F_{n+1},F_{n+1}+F_n) \\ \text{(}\gcd(a,a+b)=\gcd(a,b)\text{ for any }a,b\text{ — why?!)} &= \gcd(F_{n+1},F_n) \\ \text{(using the induction hypothesis)} &= 1 \end{aligned}$$

By induction, it follows that $\gcd(F_n,F_{n+1})=1$ for all integers $n\geqslant 1$. □

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