Definition 128. The **convergents** C_k of $[a_0; a_1, a_2, \ldots]$ are the truncated continued fractions $C_0 = a_0$, $C_1 = [a_0; a_1]$, $C_2 = [a_0; a_1, a_2]$, $C_k = [a_0; a_1, a_2, a_3]$, ...

Theorem 129. The convergents C_k of $[a_0;\ a_1,\ a_2,\ \ldots]$ always converge to a value x in the following alternating fashion

$$C_0 < C_2 < C_4 < \cdots \quad x \quad \cdots < C_5 < C_3 < C_1.$$

We simply write $x = [a_0; a_1, a_2, \ldots]$ for that value.

If the continued fraction is finite, that is $x = [a_0; a_1, a_2, \ldots a_n]$, then we only have the convergents $C_0, C_1, \ldots, C_n$.

Proof. See Theorem 15.4 in our book for full details. □

Example 130.

(a) Evaluate the first 4 convergents of $[2; 3, 2, 3, 2, \ldots]$.

Then, using the next result, compute 3 more convergents.

(b) Which number is represented by $[2; 3, 2, 3, 2, \ldots]$?

Solution.

(a) $C_0 = 2$

$C_1 = [2; 3] = 2 + \frac{1}{3} = \frac{7}{3} \approx 2.333$

$C_2 = [2; 3, 2] = 2 + \frac{1}{3 + \frac{1}{2}} = 2 + \frac{2}{7} = \frac{16}{7} \approx 2.286$

$C_3 = [2; 3, 2, 3] = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3}}} = \frac{55}{24} \approx 2.292$

Using the next result, we compute the convergents $C_n = \frac{p_n}{q_n}$ as follows:

n	-2	-1	0	1	2	3	4	5	6
a_n			2	3	2	3	2	3	2
p_n	0	1	2	7	16	55	126	433	992
q_n	1	0	1	3	7	24	55	189	433
C_n			$\frac{2}{1}$	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{55}{24}$	$\frac{126}{55}$	$\frac{433}{189}$	$\frac{992}{433}$

(b) Write $x = [2; 3, 2, 3, 2, \ldots]$. Then, $x = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \ldots}}} = 2 + \frac{1}{3 + \frac{1}{x}}$.

The equation $x = 2 + \frac{1}{3 + \frac{1}{x}}$ simplifies to $x - 2 = \frac{x}{3x + 1}$. Further simplifies to $(x - 2)(3x + 1) = x$ or $3x^2 - 6x - 2 = 0$, which has the solutions $x = \frac{6 \pm \sqrt{36 + 24}}{6} = 1 \pm \sqrt{\frac{5}{3}}$.

Since $1 + \sqrt{\frac{5}{3}} \approx 2.291$ and $1 - \sqrt{\frac{5}{3}} \approx -0.291$, we conclude that $[2; 3, 2, 3, 2, \ldots] = 1 + \sqrt{\frac{5}{3}}$.

Advanced comment. The fractions $\frac{p_n}{q_n}$ are always reduced! Can you see how to conclude that $\gcd(p_n, q_n) = 1$ from the relation $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$ (which can be proved by induction)?

We can see this relation quite nicely in the above table because $p_n q_{n-1} - p_{n-1} q_n$ is a 2×2 determinant taken from the rows containing p_n and q_n:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 7 \\ 1 & 3 \end{vmatrix} = -1, \quad \begin{vmatrix} 7 & 16 \\ 3 & 7 \end{vmatrix} = 1, \quad \begin{vmatrix} 16 & 55 \\ 7 & 24 \end{vmatrix} = -1, \quad \ldots$$

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Theorem 131. The kth convergent of the continued fraction $[a_0; a_1, a_2, \ldots]$ is

$$C_k = \frac{p_k}{q_k},$$

where p_k and q_k are characterized by

$$\begin{array}{l} p_k = a_k p_{k-1} + p_{k-2} \\ \text{with } p_{-2} = 0, \quad p_{-1} = 1 \end{array} \qquad \text{and} \qquad \begin{array}{l} q_k = a_k q_{k-1} + q_{k-2} \\ \text{with } q_{-2} = 1, \quad q_{-1} = 0 \end{array}.$$

Proof. We will prove the claim by induction on k.

First, we check the two base cases $k = 0$, $k = 1$ directly: $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$. In other words, $p_0 = a_0$, $q_0 = 1$ and $p_1 = a_0 a_1 + 1$, $q_1 = a_1$. This matches with the values from the recursion.

Next, we assume that the theorem is true for $k = 0, 1, \ldots, n$. In particular,

$$C_n = [a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}},$$

for any values $a_0, a_1, \ldots, a_n$. Note that $C_{n+1} = [a_0; a_1, a_2, \ldots, a_n, a_{n+1}] = \left[a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}\right]$. Replacing a_n with $a_n + \frac{1}{a_{n+1}}$, we therefore obtain

$$\begin{aligned} C_{n+1} = \left[a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}\right] &= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \\ &= \frac{(a_n a_{n+1} + 1) p_{n-1} + a_{n+1} p_{n-2}}{(a_n a_{n+1} + 1) q_{n-1} + a_{n+1} q_{n-2}} \\ &= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} \\ &= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}, \end{aligned}$$

which is what we had to prove. □

Example 132. Express $\frac{5}{3}$ as a simple continued fraction.

Solution. $\frac{5}{3} = 1 + \frac{2}{3} = 1 + \frac{1}{3/2} = 1 + \frac{1}{1 + \frac{1}{2}} = [1; 1, 2]$

Writing the final 2 as $1 + \frac{1}{1}$, we also have $\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = [1; 1, 1, 1]$.

More generally. If $a_n > 1$, we always have $[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_n - 1, 1]$.

Comment. Apart from these two variations, the simple continued fraction for $\frac{3}{5}$ is unique.

Note that we are used to a similar ambiguity when dealing with terminating decimal expansions: for instance, $1.25000000\ldots = 124999999\ldots$

A slight variation. It follows from the above that $\frac{3}{5} = 0 + \frac{1}{5/3} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = [0; 1, 1, 2]$.

More generally, we always have that, if $x = [a_0; a_1, a_2, \ldots]$ with $a_0 > 0$, then $\frac{1}{x} = [0; a_0, a_1, a_2, \ldots]$.

Example 133. Express $\frac{43}{19}$ as a simple continued fraction.

Solution. $\frac{43}{19} = 2 + \frac{5}{19} = 2 + \frac{1}{19/5} = 2 + \frac{1}{3 + \frac{4}{5}} = 2 + \frac{1}{3 + \frac{1}{5/4}} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}} = [2; 3, 1, 4]$

Again, also, $\frac{43}{19} = [2; 3, 1, 4] = \left[2; 3, 1, 3 + \frac{1}{1}\right] = [2; 3, 1, 3, 1]$.

Super important comment. We have done this computation before (in a different guise)!

By the Euclidian algorithm: $43 = \boxed{2} \cdot 19 + 5$, $19 = \boxed{3} \cdot 5 + 4$, $5 = \boxed{1} \cdot 4 + 1$, $4 = \boxed{4} \cdot 1 + 0$.

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