Example 21. Gauss' summation can be written as $1+2+\ldots+n=\binom{n+1}{2}$. $[\binom{n+1}{2}=\frac{n(n+1)}{2}]$

This inspires yet another (direct!) way of proving it:

Proof. We claim that both sides count the number of ways in which 2 things can be selected from $n+1$ things. For the right-hand side this is just the definition. For the left-hand side, imagine the $n+1$ things lined up in some order. We could select the first thing and then have n choices for the second. Or, we could select the second thing and then have $n-1$ choices for the third (because we now need to avoid selecting the first element again). Or, select the third things and one of the $n-2$ ones after it. In total, there are $n+(n-1)+(n-2)+\ldots+2+1$ many possibilities. □

Example 22. Likewise, $\binom{n+1}{3}=\binom{n}{2}+\binom{n-1}{2}+\ldots+\binom{2}{2}$. Can you reproduce the argument?

And, more generally, $\binom{n+1}{k+1}=\binom{n}{k}+\binom{n-1}{k}+\ldots+\binom{k}{k}$. For short, $\binom{n+1}{k+1}=\sum_{j=k}^{n}\binom{n}{j}$.

Example 23. Observe the following connection with our sums and integrals from calculus:

- $\int_0^n x\mathrm{d}x=\frac{n^2}{2}$ versus $\sum_{x=0}^{n} x=1+2+\ldots+n=\frac{n(n+1)}{2}=\frac{n^2}{2}+\text{lower order terms}$
- $\int_0^n x^2\mathrm{d}x=\frac{n^3}{3}$ versus $\sum_{x=0}^{n} x^2=1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}=\frac{n^3}{3}+\text{lower order terms}$
- $\int_0^n x^3\mathrm{d}x=\frac{n^4}{4}$ versus $\sum_{x=0}^{n} x^3=1^3+2^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2=\frac{n^4}{4}+\text{lower order terms}$

The connection makes sense: the integrals give areas below curves, and the sums are approximations to these areas (rectangles of width 1).

2 Divisibility

2.1 Quotients and remainders

Theorem 24. Let $a,b\in\mathbb{Z}$, with $b\neq 0$. Then there exist unique integers q and r such that

$$a=qb+r,\quad 0\leqslant r<|b| \qquad \text{(that is, } \tfrac{a}{b}=q+\tfrac{r}{b}\text{)}.$$

q is the **quotient**, and r the **remainder** in the division of a by b.

Example 25. For $a=20$, $b=6$, we have $\frac{20}{6}=3+\frac{2}{6}$. That is, $q=3$ and $r=2$.
For $a=20$, $b=5$, we have $\frac{20}{5}=4+\frac{0}{5}$. That is, $q=4$ and $r=0$.

Example 26. When $b=2$, then $r\in\{0,1\}$, and every integer is either of the form $2q$ or of the form $2q+1$. We call numbers **even** or **odd** correspondingly.

Example 27. Show that the square of an integer leaves the remainder 0 or 1 upon division by 4.

That is, none of the squares $1,4,9,16,25,36,\ldots$ leave remainder 2 or 3 when dividing by 4!!

Proof. Every integer is of the form $2q$ or $2q+1$. Upon division by 4, $(2q)^2=4q^2$ leaves remainder 0, $(2q+1)^2=4q^2+4q+1$ leaves remainder 1.

Example 28. Show that the square of an integer leaves the remainder 0 or 1 upon division by 3.

Proof. Every integer is of the form $3q$, $3q+1$ or $3q+2$. Upon division by 3, $(3q)^2=9q^2$ leaves remainder 0, while both $(3q+1)^2=9q^2+6q+1$ and $(3q+2)^2=9q^2+12q+4$ leave remainder 1.

Armin Straub
straub@southalabama.edu