1.3 Proofs by induction

(mathematical induction) To prove that $\mathrm{CLAIM}(n)$ is true for all integers $n \geqslant n_0$, it suffices to show:

- **(base case)** $\mathrm{CLAIM}(n_0)$ is true.
- **(induction step)** if $\mathrm{CLAIM}(n)$ is true for some n, then $\mathrm{CLAIM}(n+1)$ is true as well.

Why does this work? By the base case, $\mathrm{CLAIM}(n_0)$ is true. Thus, by the induction step, $\mathrm{CLAIM}(n_0+1)$ is true. Applying the induction step again shows that $\mathrm{CLAIM}(n_0+2)$ is true, ...

Example 5. (Gauss, again) For all integers $n \geqslant 1$, $1+2+\ldots+n = \frac{n(n+1)}{2}$.

Proof. Again, write $s(n) = 1+2+\ldots+n$.
$\mathrm{CLAIM}(n)$ is that $s(n) = \frac{n(n+1)}{2}$.

- **(base case)** $\mathrm{CLAIM}(1)$ is that $s(1) = \frac{1(1+1)}{2} = 1$. That's true.
- **(induction step)** Assume that $\mathrm{CLAIM}(n)$ is true (the **induction hypothesis**).

$$s(n+1) = s(n) + (n+1) = \underbrace{\frac{n(n+1)}{2}}_{\substack{\text{this is where we use}\\ \text{the induction hypothesis}}} + (n+1) = \frac{(n+1)(n+2)}{2}$$

This shows that $\mathrm{CLAIM}(n+1)$ is true as well.

By induction, the formula is therefore true for all integers $n \geqslant 1$. □

Comment. The claim is also true for $n = 0$ (if we interpret the left-hand side correctly).

Example 6. (sum of squares) For all integers $n \geqslant 1$, $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Write $t(n) = 1^2+2^2+\ldots+n^2$.
We use induction on the claim $t(n) = \frac{n(n+1)(2n+1)}{6}$.

- The base case $(n=1)$ is that $t(1) = 1$. That's true.
- For the inductive step, assume the formula holds for some value of n.
We need to show the formula also holds for $n+1$.

$$\begin{aligned} t(n+1) &= t(n) + (n+1)^2 \\ \text{(using the induction hypothesis)} &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)}{6}[2n^2+n+6n+6] \\ &= \frac{(n+1)}{6}(n+2)(2n+3) \end{aligned}$$

This shows that the formula also holds for $n+1$.

By induction, the formula is true for all integers $n \geqslant 1$. □

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Example 7. (a different approach to sums of powers) Let $k\in\mathbb{N}$. The preceeding two cases suggest that, in general, $p(n)=1^k+2^k+\ldots+n^k$ is a polynomial in n of degree $k+1$.

It is not hard to prove this fact, but would lead us a bit astray. Let us just assume it as fact for now (and note that we could resort to induction to prove any specific claim we are coming up with as a consequence).

Connections. These are very interesting polynomials and can be expressed as **Bernoulli polynomials**.

On the other hand, recall the following important fact:

A polynomial of degree d is uniquely determined by $d+1$ values.

Why? Such a polynomial in n, say, can be written as $c_0+c_1n+c_2n^2+\ldots+c_dn^d$. It involves $d+1$ coefficients. A special case everyone is familiar with is that a line is determined by 2 points.

We can combine these two facts, we can give much simpler proofs:

- To prove that $1+2+\ldots+n=\frac{n(n+1)}{2}$, we only need to observe that both sides are polynomials in n of degree 2, and that they take the same values for 3 different choices of n (say, $n=1$, $n=2$ and $n=3$). Indeed, for $n=1$, both sides equal $1=\frac{1\cdot 2}{2}$. For $n=2$, $3=\frac{2\cdot 3}{2}$. For $n=3$, $6=\frac{3\cdot 4}{2}$.

- Likewise, to prove that $1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$, we only need to observe that both sides are polynomials in n of degree 3, and that they take the same values for 4 different choices of n (say, $n=1$, $n=2$ and $n=3$). Check that!

On the other hand, let us turn the table around, and produce a formula for $1^3+2^3+\ldots+n^3$ in that fashion.

- In other words, we are looking for a polynomial $p(n)$ of degree 4 with the property that $p(1)=1$, $p(2)=9$, $p(3)=36$, $p(4)=100$, $p(5)=225$.

  Looks like these are all squares! Let us therefore look instead for a polynomial $q(n)$ of degree 2 with the property that $q(1)=1$, $q(2)=3$, $q(3)=6$. (Why are we only listing 3 values?)

  Here is the (unique!) such polynomial $q(n)$ (make sure you can really see that it is of degree 2 and takes the values $q(1)=1$, $q(2)=3$, $q(3)=6$ — writing down this polynomial goes by the name of **Lagrange interpolation**):

$$\begin{aligned} q(n) &= 1\frac{(n-2)(n-3)}{(1-2)(1-3)}+3\frac{(n-1)(n-3)}{(2-1)(2-3)}+6\frac{(n-1)(n-2)}{(3-1)(3-2)} \\ &= \frac{1}{2}(n-2)(n-3)-3(n-1)(n-3)+3(n-1)(n-2)=\frac{n^2+n}{2} \end{aligned}$$

  We can now verify that $p(n)=q(n)^2=\left(\frac{n(n+1)}{2}\right)^2$ is the degree 4 polynomial meeting our needs.

  In other words, we have discovered ourselves that $1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$.

Example 8. (Homework) Using induction, prove that $1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$.

Example 9. (Homework)

(a) Experiment to find a formula for $1+3+5+\ldots+(2n+1)$.

(b) Prove that formula using induction.

(c) Can you give a second proof using Gauss' result?

Example 10. (Optional homework) Can you discover the formula for $1^2+2^2+\ldots+n^2$ in the same way as we discovered the formula for sums of cubes?

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