Homework #1

MATH 311 — Intro to Number Theory
due in class on Thursday, Sep 1

Please print your name:

These problems are not suited to be done last minute!
Also, if you start early, you can consult with me if you should get stuck.

Problem 1. Using induction, prove that $1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Problem 2.

(a) Experiment to find a formula for $1 + 3 + 5 + \ldots + (2n+1)$.

(b) Prove that formula using induction.

(c) Can you give a second (direct) proof using Gauss' formula for $1 + 2 + 3 + \ldots + n$?

Problem 3. Consider the sequence a_n defined recursively by $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, and

$$a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3}.$$

(a) Compute a_7. [Your answer should be 169.]

(b) Come up with a conjecture that relates the numbers a_n with a sequence we have already seen in class.
If you get stuck, just ask me! I will be happy to give hints as needed.

(c) **(nothing to do here)** Unfortunately, it is not a simple matter to prove your conjecture with just what we know so far. However, you are certainly invited to try (and get some bonus).

Instead, we aim for lower hanging fruit: (Both claims that follow are clearly true if we could prove our conjecture.)

(d) Using (strong) induction, prove that the sequence a_n is increasing (that is, $a_n \geqslant a_{n-1}$ for all integers $n \geqslant 1$).

(e) Using (strong) induction, prove that $a_n < 4^n$ for all integers $n \geqslant 0$.
Note that, by the previous part, we now know $a_n \geqslant 0$. (Why is this **not** completely obvious from the definition?)

Caution. For the last two parts, how many base cases need to be considered?

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