Midterm #1

MATH 316 — Linear Algebra II
Friday, Feb 17, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 29 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points)

(a) Find the least squares solution to $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 2 \end{bmatrix}$.

(b) Determine the least squares line for the data points $(-1, 2), (0, 4), (1, -1), (1, 2)$.

Solution. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 2 \end{bmatrix}$.

(a) Since $A^T A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$, so the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ are

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

Solving, we find that the least squares solution is $\hat{\boldsymbol{x}} = \frac{1}{11} \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(b) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \\ 2 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Hence, the least squares line is $y = 2 - x$.

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Problem 2. (8 points)

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ onto W.

(c) Determine the QR decomposition of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution.

(a) Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be the vectors spanning W. We first construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ using Gram–Schmidt (and then normalize afterwards):

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Normalizing, we obtain the orthonormal basis $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ for W.

(b) The orthogonal projection of $\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ onto W is

$$\frac{\boldsymbol{v} \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 + \frac{\boldsymbol{v} \cdot \boldsymbol{q}_2}{\boldsymbol{q}_2 \cdot \boldsymbol{q}_2} \boldsymbol{q}_2 = \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

(Check: the error $\frac{2}{3}(1, -2, -1)^T$ is indeed orthogonal to W.)

(c) From the first part, we know that $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$.

Problem 3. (3 points) We want to find values for the parameters a, b, c such that $z = ax^2 + b + \frac{c}{y}$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution. The equations $ax_i^2 + b + \frac{c}{y_i} = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} x_1^2 & 1 & 1/y_1 \\ x_2^2 & 1 & 1/y_2 \\ x_3^2 & 1 & 1/y_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{z}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution.

Problem 4. (3 points) Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

(a) A basis for null(A) is ☐. A basis for col(A) is ☐.

(b) $\dim \operatorname{col}(A) =$ ☐, $\dim \operatorname{row}(A) =$ ☐, $\dim \operatorname{null}(A) =$ ☐, $\dim \operatorname{null}(A^T) =$ ☐

Solution.

(a) A basis for null(A) is $\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$. A basis for col($A$) is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) $\dim \operatorname{col}(A) = 2$, $\dim \operatorname{row}(A) = 2$, $\dim \operatorname{null}(A) = 2$, $\dim \operatorname{null}(A^T) = 0$

Problem 5. (9 points) Fill in the blanks.

(a) Suppose that A is a symmetric 2×2 matrix with 4-eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\det(A) = 2$.

Then A has ☐-eigenvector ☐. [Don't select a multiple of the given eigenvector.]

(b) If A is a 4×8 matrix with rank 3, then $\dim \operatorname{row}(A) =$ ☐ and $\dim \operatorname{null}(A) =$ ☐.

(c) $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ if and only if ☐.

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(d) By definition, a matrix Q is orthogonal if and only if [ ].

(e) If Q is orthogonal, then $\det(Q)$ is [ ].

(f) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to [ ].

(g) $\operatorname{col}(A)$ is the orthogonal complement of [ ]. $\operatorname{null}(A)$ is the orthogonal complement of [ ].

(h) The projection matrix for orthogonally projecting onto $\operatorname{col}(A)$ is [ ]. [A has independent columns.]

If A has orthonormal columns, this simplifies to [ ].

(i) If W is the subspace of $\mathbb{R}^7$ of all solutions to $x_1 - x_4 - 3x_7 = 0$, then $\dim W =$ [ ], $\dim W^{\perp} =$ [ ].

Solution.

(a) If A is a symmetric 2×2 matrix with 4-eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\det(A) = 2$.

Then A has $\frac{1}{2}$-eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(b) If A is a 4×8 matrix with rank 3, then $\dim \operatorname{row}(A) = 3$ and $\dim \operatorname{null}(A) = 8 - 3 = 5$.

(c) $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$.

(d) By definition, a matrix Q is orthogonal if and only if Q is $n \times n$ (square) and Q has orthonormal columns.

(e) If Q is orthogonal, then $\det(Q)$ is ± 1.

(f) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$.

(g) $\operatorname{col}(A)$ is the orthogonal complement of $\operatorname{null}(A^T)$. $\operatorname{null}(A)$ is the orthogonal complement of $\operatorname{row}(A)$.

(h) The projection matrix for orthogonally projecting onto $\operatorname{col}(A)$ is $P = A\,(A^T A)^{-1} A^T$.

If A has orthonormal columns (so that $A^T A = I$), this simplifies to $A A^T$.

If A is orthogonal, this further simplifies to I.

(i) $\dim W = 7 - 1 = 6$ and $\dim W^{\perp} = 7 - 6 = 1$.

(extra scratch paper)