Example 179. Find the best approximation of $f(x) = \sqrt{x}$ on the interval $[0, 1]$ using a function of the form $y = a + bx$.

Important observation. The orthogonal projection of $f: [0,1] \to \mathbb{R}$ onto $\operatorname{span}\{1, x\}$ is not simply the projection onto 1 plus the projection onto x. That's because 1 and x are not orthogonal:

$$\langle 1, x \rangle = \int_0^1 t \mathrm{d}t = \frac{1}{2} \neq 0.$$

Solution. To find an orthogonal basis for $\operatorname{span}\{1, x\}$, following Gram–Schmidt, we compute

$$x - \left(\begin{array}{c} \text{projection of} \\ x \text{ onto } 1 \end{array} \right) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{1}{2}.$$

Hence, $1, x - \frac{1}{2}$ is an orthogonal basis for $\operatorname{span}\{1, x\}$.
The orthogonal projection of $\sqrt{x}$ on $[0, 1]$ onto $\operatorname{span}\{1, x\} = \operatorname{span}\left\{1, x - \frac{1}{2}\right\}$ therefore is

$$\frac{\langle \sqrt{x}, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\left\langle \sqrt{x}, x - \frac{1}{2} \right\rangle}{\left\langle x - \frac{1}{2}, x - \frac{1}{2} \right\rangle} \left(x - \frac{1}{2} \right) = \frac{\int_0^1 \sqrt{t} \mathrm{d}t}{\int_0^1 1 \mathrm{d}t} + \frac{\int_0^1 \sqrt{t} \left(t - \frac{1}{2} \right) \mathrm{d}t}{\int_0^1 \left(t - \frac{1}{2} \right)^2 \mathrm{d}t} \left(x - \frac{1}{2} \right).$$

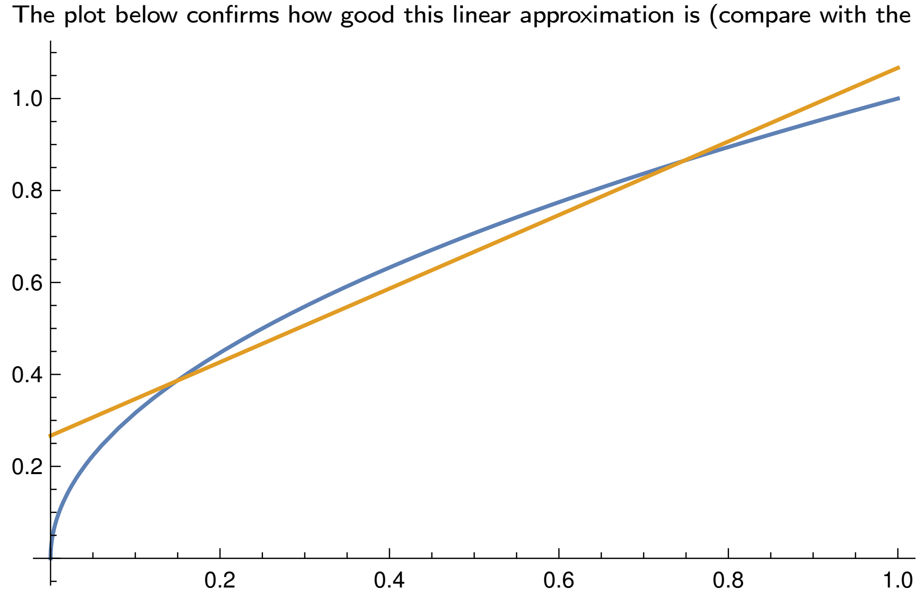
We compute the three new integrals:

$$\begin{aligned}
\int_0^1 \sqrt{t} \mathrm{d}t &= \left[\frac{2}{3} t^{3/2} \right]_0^1 = \frac{2}{3} \\
\int_0^1 \sqrt{t} \left(t - \frac{1}{2} \right) \mathrm{d}t &= \int_0^1 \left(t^{3/2} - \frac{1}{2} t^{1/2} \right) \mathrm{d}t = \left[\frac{2}{5} t^{5/2} - \frac{1}{3} t^{3/2} \right]_0^1 = \frac{2}{5} - \frac{1}{3} = \frac{1}{15} \\
\int_0^1 \left(t - \frac{1}{2} \right)^2 \mathrm{d}t &= \int_0^1 \left(t^2 - t + \frac{1}{4} \right) \mathrm{d}t = \left[\frac{1}{3} t^3 - \frac{1}{2} t^2 + \frac{1}{4} t \right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}
\end{aligned}$$

Using these values, the best approximation is

$$\frac{\int_0^1 \sqrt{t} \mathrm{d}t}{\int_0^1 1 \mathrm{d}t} + \frac{\int_0^1 \sqrt{t} \left(t - \frac{1}{2} \right) \mathrm{d}t}{\int_0^1 \left(t - \frac{1}{2} \right)^2 \mathrm{d}t} \left(x - \frac{1}{2} \right) = \frac{2}{3} + \frac{12}{15} \left(x - \frac{1}{2} \right) = \frac{4}{5} x + \frac{4}{15}$$

The plot below confirms how good this linear approximation is (compare with the previous example):



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