Review. A matrix A has orthonormal columns $\iff$ $A^T A = I$.

Example 76. Suppose Q has orthonormal columns. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(Q)$?

Solution. Recall that, to project onto $\mathrm{col}(A)$, the projection matrix is $P = A(A^T A)^{-1} A^T$.

Since $Q^T Q = I$, to project onto $\mathrm{col}(Q)$, the projection matrix is $P = Q Q^T$.

Comment. A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q}\cdot\boldsymbol{b})\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}^T\boldsymbol{b}) = (\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.

Comment. In particular, if Q is not square, then $Q^T Q = I$ but $Q Q^T \neq I$. In some sense, $Q Q^T$ still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\mathrm{col}(Q)$ it does act like the identity for vectors in $\mathrm{col}(Q)$. (Vectors not in $\mathrm{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\mathrm{col}(Q)$.)

Example 77. Suppose A is invertible. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(A)$?

Solution. If A is an invertible $n\times n$ matrix, then $\mathrm{col}(A) = \mathbb{R}^n$ (because the n columns of A are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\mathrm{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

Definition 78. An **orthogonal matrix** is a square matrix with orthonormal columns.

[This is not a typo (but a confusing convention): the columns need to be orthonormal, not just orthogonal.]

An $n\times n$ matrix Q is orthogonal $\iff$ $Q^T Q = I$

In other words, $Q^{-1} = Q^T$.

Review. Recall the following properties of determinants:

- $\det(AB) = \det(A)\det(B)$

  Comment. In fancy language, this means that the determinant is a group homomorphism between the group of (invertible) $n\times n$ matrices and (nonzero) complex numbers. Note that, on the left hand, we are multiplying matrices while, on the right hand, we are multiplying numbers. The key point is that it doesn't matter which multiplication we do: the two multiplications are compatible.

- $\det(A^{-1}) = \frac{1}{\det(A)}$

  Comment. Can you derive this from the previous property?

- $\det(A^T) = \det(A)$

  Comment. We are familiar with this in the context of cofactor expansion: it doesn't matter whether we expand by a column or by a row.

Example 79. What can we say about $\det(Q)$ if Q is orthogonal?

Solution. Write $d = \det(Q)$. Since $Q^{-1} = Q^T$, we have $\frac{1}{d} = d$ (recall that $\det(Q^{-1}) = 1/\det(Q)$ and $\det(Q^T) = \det(Q)$) or, equivalently, $d^2 = 1$. Hence, $d = \pm 1$.

Both of these are possible as the examples $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ illustrate.

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Review: Diagonalizability

Example 80. (review) If A is a 2×2 matrix with $\det(A) = -8$ and eigenvalue 4. What is the second eigenvalue?

Solution. Recall that $\det(A)$ is the product of the eigenvalues (see below). Hence, the second eigenvalue is -2.

$\det(A)$ is the product of the eigenvalues of A.

Why? Recall how we determine the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n\times n$ matrix A. We compute the characteristic polynomial $\det(A-\lambda I)$ and determine the λ_i as the roots of that polynomial.

That means that we have the factorization $\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_n-\lambda)\cdots(\lambda_n-\lambda)$. Now, set $\lambda=0$ to conclude that $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

Lemma 81. A matrix A is diagonalizable if and only if, for every eigenvalue λ that is k times repeated, the λ-eigenspace of A has dimension k.

In short, an $n\times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e. "there are enough eigenvectors").

The next two examples illustrate that not all matrices are diagonalizable and that, even if a real matrix is diagonalizable, the eigenvalues and eigenvectors might be complex.

Example 82. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2$, which has $\lambda = 0$ as a double root.

However, the 0-eigenspace $\operatorname{null}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is only 1-dimensional.

As a consequence, A is not diagonalizable.

Example 83. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2+1 = (\lambda-i)(\lambda+i)$.

Hence, the eigenvalues are $\pm i$.

The i-eigenspace $\operatorname{null}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right)$ has basis $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

The $-i$-eigenspace $\operatorname{null}\left(\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\right)$ has basis $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Thus, A has the diagonalization $A = PDP^{-1}$ with $D = \begin{bmatrix} i & \\ & -i \end{bmatrix}$ and $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$.

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