Example 23. (review) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Recall that $(AB)^T = B^T A^T$. This reflects the fact that, in the column-centric versus the row-centric interpretation of matrix multiplication, the order of the matrices is reversed.

Comment. When working with complex numbers, the fundamental role is not played by the transpose but by the **conjugate transpose** instead (we'll see that in our discussion of orthogonality): $A^* = \overline{A^T}$.

For instance, if $A = \begin{bmatrix} 1-3i & 5i \\ 2+i & 3 \end{bmatrix}$, then $A^* = \begin{bmatrix} 1+3i & 2-i \\ -5i & 3 \end{bmatrix}$.

Orthogonality

The inner product and distances

Definition 24. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \; = \; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 25. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

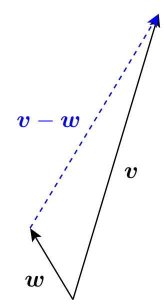
Definition 26.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \; = \; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \; = \; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 27. For instance, in $\mathbb{R}^2$, $\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 28. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

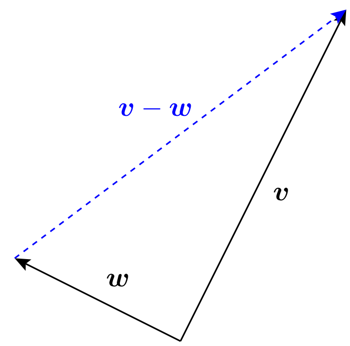
Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

Comment. This is a vector version of $(x-y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

Definition 29. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v}\cdot\boldsymbol{w}=0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$\iff \|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2=\underbrace{\|\boldsymbol{v}-\boldsymbol{w}\|^2}_{\substack{=\|\boldsymbol{v}\|^2-2\boldsymbol{v}\cdot\boldsymbol{w}+\|\boldsymbol{w}\|^2\\ \text{(by previous example)}}}$

$\iff -2\boldsymbol{v}\cdot\boldsymbol{w}=0$

$\iff \boldsymbol{v}\cdot\boldsymbol{w}=0$

Example 30. Determine a basis for the **orthogonal complement** of (the span of) $\begin{bmatrix}1\\2\\1\end{bmatrix}$.

What are we looking for? The orthogonal complement of $\boldsymbol{v}$ consists of all vectors that are orthogonal to $\boldsymbol{v}$. More generally, the orthogonal complement of a space V consists of all vectors that are orthogonal to every vector in V.

Solution. (staring/intution) We are working in 3-dimensional space and already have 1 vector. The vectors orthogonal to it lie in a $3-1=2$-dimensional space (a plane).

Two of the vectors orthogonal to $\begin{bmatrix}1\\2\\1\end{bmatrix}$ are $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ (but there are many other choices).

Knowing that the orthogonal complement has dimension 2, we conclude that $\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}$ is a basis.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ is $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

[Note how the dimensions add up to the dimension of the entire space: $1+2=3$.]

Solution. (professional) $\begin{bmatrix}1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ (dot product!) is the same as $\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ (matrix product!).

Hence, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ is the same as $\operatorname{null}(\begin{bmatrix}1&2&1\end{bmatrix})$.

Computing a basis for $\operatorname{null}(\begin{bmatrix}1&2&1\end{bmatrix})$ is easy since $\begin{bmatrix}1&2&1\end{bmatrix}$ is already in RREF.

Note that the general solution to $\begin{bmatrix}1&2&1\end{bmatrix}\boldsymbol{x}=0$ is $\begin{bmatrix}-2s-t\\s\\t\end{bmatrix}=s\begin{bmatrix}-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

A basis for $\operatorname{null}(\begin{bmatrix}1&2&1\end{bmatrix})$ therefore is $\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}$. (Check that these are indeed orthogonal to $\begin{bmatrix}1\\2\\1\end{bmatrix}$!)

Armin Straub
straub@southalabama.edu

The fundamental theorem

Review. The four **fundamental subspaces** associated with a matrix A are

$$\mathrm{col}(A), \quad \mathrm{row}(A), \quad \mathrm{null}(A), \quad \mathrm{null}(A^T).$$

Note that $\mathrm{row}(A) = \mathrm{col}(A^T)$. (In particular, we usually write vectors in $\mathrm{row}(A)$ as column vectors.)

Comment. $\mathrm{null}(A^T)$ is called the **left null space** of A.

Why that name? Recall that, by definition $\boldsymbol{x}$ is in $\mathrm{null}(A) \iff A\boldsymbol{x} = \boldsymbol{0}$.

Likewise, $\boldsymbol{x}$ is in $\mathrm{null}(A^T) \iff A^T\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x}^T A = \boldsymbol{0}$.

[Recall that $(AB)^T = B^T A^T$. In particular, $(A^T\boldsymbol{x})^T = \boldsymbol{x}^T A$, which is what we used in the last equivalence.]

Review. The **rank** of a matrix is the number of pivots in its RREF.

Equivalently, as showcased in the next result, the rank is the dimension of either the column or the row space.

Theorem 31. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of **rank** r.

- $\dim \mathrm{col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \mathrm{row}(A) = r$ (subspace of $\mathbb{R}^n$) $\qquad \mathrm{row}(A) = \mathrm{col}(A^T)$
- $\dim \mathrm{null}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{null}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Example 32. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Solution. Make sure that, for such a simple matrix, you can see all of these that at a glance!

$\mathrm{col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$, $\mathrm{row}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, $\mathrm{null}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$, $\mathrm{null}(A^T) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Important observation. The basis vectors for $\mathrm{row}(A)$ and $\mathrm{null}(A)$ are orthogonal! $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Always. Vectors in $\mathrm{null}(A)$ are orthogonal to vectors in $\mathrm{row}(A)$. In short, $\mathrm{null}(A)$ is orthogonal to $\mathrm{row}(A)$.

Why? Suppose that $\boldsymbol{x}$ is in $\mathrm{null}(A)$. That is, $A\boldsymbol{x} = \boldsymbol{0}$. But think about what $A\boldsymbol{x} = \boldsymbol{0}$ means (row-product rule).

It means that the inner product of every row with $\boldsymbol{x}$ is zero. Which implies that $\boldsymbol{x}$ is orthogonal to the row space.

Definition 33. As done in the observation above, we say that two subspaces V and W of $\mathbb{R}^n$ are **orthogonal** if and only if every vector in V is orthogonal to every vector in W.

The **orthogonal complement** of W is the space $W^\perp$ of all vectors that are orthogonal to W.

Exercise. Show that the orthogonal complement is indeed a vector space.

Armin Straub
straub@southalabama.edu

Example 34. Suppose that V is spanned by 3 linearly independent vectors in $\mathbb{R}^5$. Determine the dimension of V and its orthogonal complement $V^\perp$.

Solution. $\dim V = 3$ and $\dim V^\perp = 5 - 3 = 2$

Theorem 35. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\operatorname{null}(A)$ is orthogonal to $\operatorname{row}(A)$. (both subspaces of $\mathbb{R}^n$)

Note that $\dim \operatorname{null}(A) + \dim \operatorname{row}(A) = n$. Hence, the two spaces are orthogonal complements.

- $\operatorname{null}(A^T)$ is orthogonal to $\operatorname{col}(A)$.

Again, the two spaces are orthogonal complements. (This is just the first part with A replaced by A^T.)

Example 36. Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$. Check that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & -3 & -9 \end{bmatrix} \overset{R_3 - \frac{3}{2}R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{-\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 - R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$, $\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

$\operatorname{null}(A)$ and $\operatorname{row}(A)$ are indeed orthogonal, as certified by:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$$

In fact, $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements because the dimensions add up to $2 + 2 = 4$.

In particular, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ form a basis of all of $\mathbb{R}^4$.

Just to make sure. Because $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is orthogonal to both basis vectors, it is orthogonal to every vector in $\operatorname{row}(A)$.

Vectors in $\operatorname{row}(A)$ are of the form $\boldsymbol{v} = a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$. Then, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \boldsymbol{v} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$

Example 37. **(extra)** Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ (as well as $\operatorname{null}(A^T)$ and $\operatorname{col}(A)$) are orthogonal complements.

Partial solution. One can almost see that $\operatorname{rank}(A) = 3$. Hence, the dimensions of the fundamental subspaces are ...

Armin Straub
straub@southalabama.edu