

## Review: Eigenvalues and eigenvectors

If  $A\mathbf{x} = \lambda\mathbf{x}$  (and  $\mathbf{x} \neq \mathbf{0}$ ), then  $\mathbf{x}$  is an **eigenvector** of  $A$  with **eigenvalue**  $\lambda$  (just a number).

Note that for the equation  $A\mathbf{x} = \lambda\mathbf{x}$  to make sense,  $A$  needs to be a square matrix (i.e.  $n \times n$ ).

Key observation:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ \iff A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ \iff (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

This homogeneous system has a nontrivial solution  $\mathbf{x}$  if and only if  $\det(A - \lambda I) = 0$ .

To find eigenvectors and eigenvalues of  $A$ :

(a) First, find the eigenvalues  $\lambda$  by solving  $\det(A - \lambda I) = 0$ .

$\det(A - \lambda I)$  is a polynomial in  $\lambda$ , called the **characteristic polynomial** of  $A$ .

(b) Then, for each eigenvalue  $\lambda$ , find corresponding eigenvectors by solving  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ .

More precisely, we find a basis of eigenvectors for the  $\lambda$ -**eigenspace**  $\text{null}(A - \lambda I)$ .

**Example 16.**  $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$  has one eigenvector that is “easy” to see. Do you see it?

**Solution.** Note that  $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . Hence,  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is a 2-eigenvector.

**Just for contrast.** Note that  $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Hence,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is not an eigenvector.

Suppose that  $A$  is  $n \times n$  and has independent eigenvectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$ .

Then  $A$  can be **diagonalized** as  $A = PDP^{-1}$ , where

- the columns of  $P$  are the eigenvectors, and
- the diagonal matrix  $D$  has the eigenvalues on the diagonal.

Such a diagonalization is possible if and only if  $A$  has enough (independent) eigenvectors.

**Comment.** If you don't quite recall why these choices result in the diagonalization  $A = PDP^{-1}$ , note that the diagonalization is equivalent to  $AP = PD$ .

- Put the eigenvectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  as columns into a matrix  $P$ .

$$\begin{aligned} A\mathbf{x}_i = \lambda_i\mathbf{x}_i \implies A \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & & | \end{bmatrix} &= \begin{bmatrix} | & & | \\ \lambda_1\mathbf{x}_1 & \dots & \lambda_n\mathbf{x}_n \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \end{aligned}$$

- In summary:  $AP = PD$

**Example 17.** Let  $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ .

- Find the eigenvalues and bases for the eigenspaces of  $A$ .
- Diagonalize  $A$ . That is, determine matrices  $P$  and  $D$  such that  $A = PDP^{-1}$ .

**Solution.**

- By expanding by the second column, we find that the characteristic polynomial  $\det(A - \lambda I)$  is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda) - 2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are  $\lambda = 2$  (with multiplicity 2) and  $\lambda = 5$ .

**Comment.** At this point, we know that we will find one eigenvector for  $\lambda = 5$  (more precisely, the 5-eigenspace definitely has dimension 1). On the other hand, the 2-eigenspace might have dimension 2 or 1. In order for  $A$  to be diagonalizable, the 2-eigenspace must have dimension 2. (Why?!)

- The 5-eigenspace is  $\text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right)$ . Proceeding as in Example 14, we obtain

$$\text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \stackrel{\text{RREF}}{=} \text{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}.$$

In other words, the 5-eigenspace has basis  $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ .

- The 2-eigenspace is  $\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$ . Proceeding as in Example 15, we obtain

$$\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \stackrel{\text{RREF}}{=} \text{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

In other words, the 2-eigenspace has basis  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ .

**Comment.** So, indeed, the 2-eigenspace has dimension 2. In particular,  $A$  is diagonalizable.

- A possible choice is  $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ ,  $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

**Comment.** However, many other choices are possible and correct. For instance, the order of the eigenvalues in  $D$  doesn't matter (as long as the same order is used for  $P$ ). Also, for  $P$ , the columns can be chosen to be any other set of eigenvectors.