

**Example 7.** Let us do Gaussian elimination on  $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$  until we have an echelon form:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \Rightarrow R_2} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

As last class, the row operation can be encoded by multiplication with an “almost identity matrix”  $E$ :

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_E \underbrace{\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}}_U$$

Since  $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$  (no calculation needed; this is the row operation  $R_2 + 2R_1 \Rightarrow R_2$  which reverses our above operation), this means that

$$A = E^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}.$$

We factored  $A$  as the product of a lower and an upper triangular matrix!

$A = LU$  is known as the **LU decomposition** of  $A$ .  
 $L$  is lower triangular,  $U$  is upper triangular.

If  $A$  is  $m \times n$ , then  $L$  is an invertible lower triangular  $m \times m$  matrix, and  $U$  is a usual echelon form of  $A$ . Every matrix  $A$  has a LU decomposition (after possibly swapping some rows of  $A$  first).

- The matrix  $U$  is just the echelon form of  $A$  produced during Gaussian elimination.
- The matrix  $L$  can be constructed, entry-by-entry, by simply recording the row operations used during Gaussian elimination. (No extra work needed!)

**Example 8.** Determine the LU decomposition of  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ .

**Solution.**  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1 \Rightarrow R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$  translates into  $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ .

Since  $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$  (no calculation needed!), we therefore have  $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ .

**Example 9.** Determine the LU decomposition of  $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix}$ .

**Solution.** We perform Gaussian elimination until we arrive at an echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - 3R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 8 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 + 8R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 9 & -5 \end{bmatrix}$$

Observe that we can reverse both of these steps using the row operations  $R_2 + 3R_1 \Rightarrow R_2$  and  $R_3 - 2R_1 - 8R_2 \Rightarrow R_3$ .

Encoding these in  $L$ , the corresponding LU decomposition of  $A$  is

$$A = LU = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ -2 & -8 & 1 & \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ & -1 & 1 & -1 \\ & & 9 & -5 \end{bmatrix}.$$

Note that no further computation was required to obtain  $L$ . (The entries in the matrix  $L$  are precisely the (negative) coefficients in the original row operations.)

**Comment.** By contrast, combining the operations  $R_2 - 3R_1 \Rightarrow R_2$  and  $R_3 + 8R_2 \Rightarrow R_3$  requires computation.

That is because we change  $R_2$  in the first step, and then use the changed  $R_2$  in the second step. Indeed, note that

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -3 & 1 & \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ -22 & 8 & 1 \end{bmatrix},$$

so the combined operations are  $R_2 - 3R_1 \Rightarrow R_2$  and  $R_3 - 22R_1 + 8R_2 \Rightarrow R_3$  (you can also see that directly from the operations).

On the other hand, there was no such complication when combining the reversed operations:

Combining  $R_3 - 8R_2 \Rightarrow R_3$  and  $R_2 + 3R_1 \Rightarrow R_2$  simply results in  $R_2 + 3R_1 \Rightarrow R_2$  and  $R_3 - 2R_1 - 8R_2 \Rightarrow R_3$ , as used above.

The difference is that, here, we change  $R_3$  in the first step but then don't use the changed  $R_3$  in the second step. In terms of matrix multiplication, we have

$$\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -2 & -8 & 1 \end{bmatrix},$$

where, because of their special form, the product of the two lower triangular matrices is just "putting together" the entries (unlike in the non-reversed product).

**Review.** The RREF (row-reduced echelon form) of  $A$  is obtained from an echelon form by

- scaling the pivots to 1, and then
- eliminating the entries above the pivots.

A typical RREF has the shape

[\* represents an entry that could be anything]

$$\begin{bmatrix} 1 & * & 0 & * & * & 0 & * \\ & & 1 & * & * & 0 & * \\ & & & & & 1 & * \end{bmatrix}$$

**Example 10.** Let's compute the RREF of the  $3 \times 4$  matrix from Example 9.

**Solution.**

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 7 & 2 \\ -2 & 6 & -3 & 1 \end{bmatrix} \xrightarrow[\sim]{\begin{matrix} R_2 - 3R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3 \end{matrix}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 8 & 1 & 3 \end{bmatrix} \xrightarrow[\sim]{R_3 + 8R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 9 & -5 \end{bmatrix}$$

$$\xrightarrow[\sim]{\begin{matrix} -R_2 \Rightarrow R_2 \\ \frac{1}{9}R_3 \Rightarrow R_3 \end{matrix}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix} \xrightarrow[\sim]{\begin{matrix} R_1 - 2R_3 \Rightarrow R_1 \\ R_2 + R_3 \Rightarrow R_2 \end{matrix}} \begin{bmatrix} 1 & 1 & 0 & \frac{19}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix} \xrightarrow[\sim]{R_1 - R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{5}{9} \end{bmatrix}$$

**Example 11.** The RREF of  $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$  from earlier is the  $2 \times 2$  identity matrix.

**Comment.** That's not surprising: A square matrix is invertible if and only if its RREF is the identity matrix. If that isn't obvious to you, think about how you invert a matrix using Gaussian elimination (reviewed next).