Example 88. (warmup) Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

- What are the eigenspaces?
- What are A^{-1} and A^{100}?

Solution.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 3-eigenvector. In other words, the 2-eigenspace is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and the 3-eigenspace is $\operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.
- $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ and $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$

Comment. Algebraically, this looks like a very simple map. However, notice that it is not so easy to say what happens to, say, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ geometrically. That is because two things are happening: part of that vector is scaled by 2, the other part is scaled by 3.

Example 89. If A has λ-eigenvector $\boldsymbol{v}$, then what can we say about A^2?

Solution. A^2 has λ^2-eigenvector $\boldsymbol{v}$.
[Indeed, $A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v}$. This is even easier in words: multiplying $\boldsymbol{v}$ with A has the effect of scaling it by λ; hence, multiplying it with A^2 scales it by λ^2.]
Important comment. Similarly, A^{100} has λ^{100}-eigenvector $\boldsymbol{v}$.

Example 90. If a matrix A can be diagonalized as $A = PDP^{-1}$, what can we say about A^n?

Solution. First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$. Likewise, $A^n = PD^nP^{-1}$.
[The point being that D^n is trivial to compute because D is diagonal.]
In particular. $A^{-1} = PD^{-1}P^{-1}$

Example 91. Let $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$. Compute A^n.

Solution. First, we diagonalize: $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 10 & \\ & 5 \end{bmatrix}$. (Fill in the details!)

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 10^n & \\ & 5^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 10^n & 10^n \\ -4 \cdot 5^n & 1 \cdot 5^n \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10^n + 4 \cdot 5^n & 10^n - 5^n \\ 4 \cdot 10^n - 4 \cdot 5^n & 4 \cdot 10^n + 5^n \end{bmatrix}$$

Check. Verify the cases $n = 0$ ($A^0 = I$) and $n = 1$.

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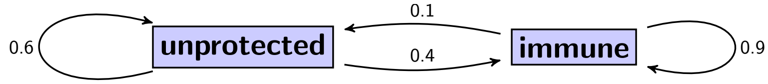
Application: Markov chains

Example 92. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.

What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)

y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

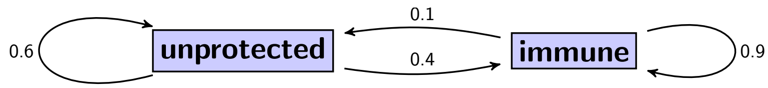
It follows that M^2 describes the transition over 2 years. Likewise, M^n describes the transition over n years.

In particular, $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. Therefore, the powers of M are the key to understanding what happens in this model over time.

Example 93. (cont'd) Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.

What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)

y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$. In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{1+4} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$.

Hence, the immunization rate in the long term equilibrium is $4/5 = 80\%$.

[Ponder about why this is a reasonable value!]

Comment. What's the other eigenvalue of the transition matrix? No need to compute the characteristic polynomial: we can easily see that it is $0.5 = 0.6 \cdot 0.9 - 0.1 \cdot 0.4$ because the product of the eigenvalues equals the determinant!

The 0.5-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Comment. Will the immunization rate always stabilize and approach the long term equilibrium? Yes! This is a consequence of the other eigenvalue of the transition matrix satisfying $|0.5| < 1$. If we start in state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1^n \cdot a \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (0.5)^n \cdot b \begin{bmatrix} -1 \\ 1 \end{bmatrix} \underset{\text{as } n \to \infty}{\longrightarrow} a \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Random comment. A rule of thumb is that a tetanus vaccination begins to wear off after about 10 years (somewhat in line with the 0.1 transition proportion in this example). However, the tetanus immunzation rate in the United States appears to be considerable less than the 80% we found in this (awfully simplistic) example.

https://www.cdc.gov/mmwr/preview/mmwrhtml/mm5940a3.htm

Example 94. Compute M^n for $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$.

Solution. In Example 91, we computed that $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$ had powers $A^n = \frac{1}{5} \begin{bmatrix} 10^n + 4 \cdot 5^n & 10^n - 5^n \\ 4 \cdot 10^n - 4 \cdot 5^n & 4 \cdot 10^n + 5^n \end{bmatrix}$.

Since $M = \frac{1}{10} A$, this implies that $M^n = \frac{1}{10^n} A^n = \frac{1}{5} \begin{bmatrix} 1 + 4 \cdot 0.5^n & 1 - 0.5^n \\ 4 - 4 \cdot 0.5^n & 4 + 0.5^n \end{bmatrix}$.

Note that $M^n \to \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$ as $n \to \infty$. This reflects the fact that $\frac{1}{5} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the long term equilibrium.

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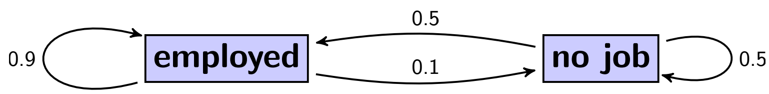
Example 95. (extra) Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Solution. Let x_t and y_t be the proportions of those employed and unemployed. Proceeding, as in the previous example, the transition matrix is $M = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$.



The 1-eigenspace of M, that is $\operatorname{null}\left(\begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix}\right)$, has basis $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$. The corresponding equilibrium is $\frac{1}{5+1}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

In particular, the unemployment rate in the long term equilibrium is $1/6 \approx 16.7\%$.

Example 96. Which of the following are true for all square matrices A?

- Is it true that A^T has the same eigenvalues as A?
- Is it true that A^T has the same eigenspaces as A?
- Is it true that A^T has the same characteristic polynomial as A?

Solution. True. False. True.

First, note that the characteristic polynomial $\det(A - \lambda I)$ is the same as $\det(A^T - \lambda I)$. [Make sure you can fill in the details of why this is the case!] Hence, the eigenvalues (which are the roots of the characteristic polynomial) are also the same for A and A^T.

On the other hand, A^T and A in general have very different eigenspaces. Take, for instance, the matrix $A = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ from Example 93. Then both A and A^T have eigenvalues $\lambda = 0.5, 1$.

However, the 1-eigenspace of A is spanned by $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, while the 1-eigenspace of A^T is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 97. Show that a Markov matrix A (so that the columns of A sum to 1) always has eigenvalue 1.

Solution. This follows because the transpose A^T always has $\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$ as an 1-eigenvector (by virtue of the rows of A^T summing to 1). [Make sure that makes sense!]

By the previous example, A must also have eigenvalue 1 (but we have no idea what a 1-eigenvector is until we compute it).

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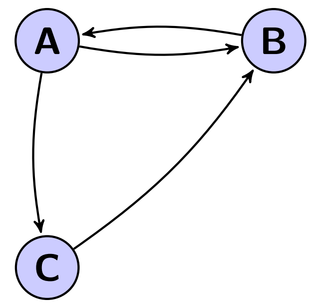
Application: PageRank

Example 98. Suppose the internet consists of only the three webpages A, B, C.

We wish to rank these webpages in order of "importance".



The idea. Instead of analyzing each webpage (which would be a lot of work!) we will try to only use the information how the pages are linked to each other. The idea being that an "important" page should be linked to from many other pages.

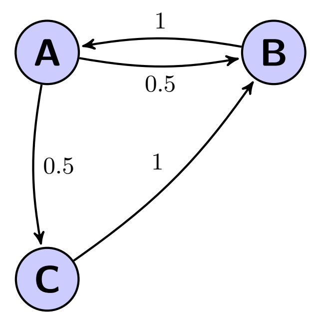
A and B have a link to each other. Also, A links to C and C links to B. If you keep randomly clicking from one webpage to the next, what proportion of the time will you be at each page?

The idea. We will assign ranking to the pages according to how frequently such a random surfer would visit these pages.

Comment. Before we start computing, stop for a moment, and think about how you would rank the webpages.

Solution. Let a_t be the probability that we will be on page A at time t. Likewise, b_t, c_t are the probabilities that we will be on page B or C.

The transition from one state to the next now works exactly as in the previous example. We get the following transition matrix:



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + 1 \cdot b_t + 0 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 1 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

To find the equilibrium state, we again determine an appropriate 1-eigenvector.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix}\right)$ which has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. In this context, this is also known as the **PageRank vector**.

In other words, after browsing randomly for a long time, there is (about) a $\frac{2}{5} = 40\%$ chance to be at page A, a $\frac{2}{5} = 40\%$ chance to be at page B, and a $\frac{1}{5} = 20\%$ chance to be at page C.

We therefore rank A and B highest (tied), and C lowest.

Just checking. Maybe we were expecting B to be ranked above A, because B is the only page that has two incoming links. However, if we are at page B, then our next click will be to page A, which is why A and B receive equal ranking.

This method of ranking is the famous **PageRank** algorithm (underlying Google's search algorithm).

By the way, the algorithm is named, not after ranking web"pages", but after Larry Page (who founded Google in 1998 together with Sergey Brin).

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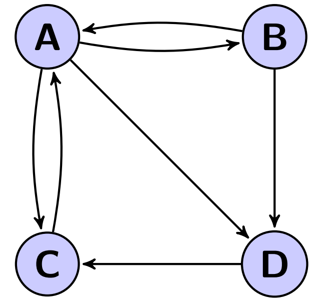
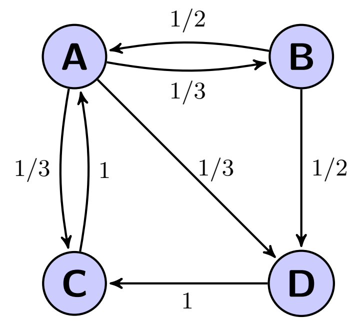
Example 99. Suppose the internet consists of only the four webpages A, B, C, D which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Solution. Recall that we model a random surfer, who randomly clicks on links. Let a_t be the probability that such a surfer will be on page A at time t. Likewise, b_t, c_t, d_t are the probabilities that the surfer will be on page B, C or D.

The transition probabilities are indicated in the diagram to the right. As in the previous example, we obtain the following transition behaviour:



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + \frac{1}{2} \cdot b_t + 1 \cdot c_t + 0 \cdot d_t \\ \frac{1}{3} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ \frac{1}{3} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 1 \cdot d_t \\ \frac{1}{3} \cdot a_t + \frac{1}{2} \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}}_{=T} \begin{bmatrix} a_t \\ b_t \\ c_t \\ d_t \end{bmatrix}$$

To find the equilibrium state, we determine an appropriate 1-eigenvector of the transition matrix T.

The 1-eigenspace is $\operatorname{null}(T - 1 \cdot I) = \operatorname{null}\left(\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix}\right)$.

To compute a basis, we perform Gaussian elimination: $\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We conclude that the 1-eigenspace has basis $\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$. (Note that its entries add up to $2 + \frac{2}{3} + \frac{5}{3} + 1 = \frac{16}{3}$.)

The corresponding equilibrium state is $\frac{3}{16}\begin{bmatrix} 2 \\ \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$. This is the **PageRank vector**.

[For instance, after browsing randomly for a long time, there is (about) a 12.5% chance to be at page B.]
Correspondingly, we rank the pages as $A > C > D > B$.

The real internet. [Google is getting more secretive about this kind of data, so the numbers are estimates from a while ago.]

- Google reports (2016) doing "trillions" of searches per year. [2 trillion means 63,000 searches per second.]
- Google's search index contains almost 50 billion pages (2016). [Estimated to exceed 100,000,000 gigabytes.]
- More than 1,000,000,000 websites (i.e. hostnames; about 75% not active)

[The "average" user apparently only visits about 100 websites per month; wikipedia.org is one website, consisting of many webpages (more than 2,000,000).]

Gory details. (extra) There's nothing interesting about the Gaussian elimination above. Here are the full details:

$$\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \underset{\substack{R_2+\frac{1}{3}R_1 \Rightarrow R_2 \\ R_3+\frac{1}{3}R_1 \Rightarrow R_3 \\ R_4+\frac{1}{3}R_1 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & -\frac{2}{3} & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \underset{\substack{R_3+\frac{1}{5}R_2 \Rightarrow R_3 \\ R_4+\frac{4}{5}R_2 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{3}{5} & 1 \\ 0 & 0 & \frac{3}{5} & -1 \end{bmatrix}$$

$$\underset{R_4+R_3 \Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{3}{5} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{-1R_1 \Rightarrow R_1 \\ -\frac{6}{5}R_2 \Rightarrow R_2 \\ -\frac{5}{3}R_3 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{R_1+R_3 \Rightarrow R_1 \\ R_2+\frac{2}{5}R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1+\frac{1}{2}R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Practical comment. The transition matrix we would get for the entire internet indexed by Google is prohibitingly large (a 50 billion by 50 billion matrix). While gigantic in size, it is a very **sparse matrix**, meaning that almost all of its entries are zero (each column has 50 billion entries but only a handful are nonzero, namely those corresponding to a link to another webpage). This is typical for many applications in linear algebra: we often deal with big but sparse matrices.

Another practical comment. It's not an issue in our simple example, but what if our random surfer gets stuck on a webpage without links? Or, similarly, gets stuck in a loop of links? To deal with these, it is customary to include "teleportation". That is, each time, one of two things happens: with probability p (typically, something like $p = 0.85$) our surfer clicks a link as before; otherwise, with probability $1-p$, he is teleported to some unrelated other page. Further, if the surfer comes to a page without links, he would teleport away.

A final practical comment. In practical situations, the system might be too large for finding the equilibrium vector by elimination, as we did above. An alternative to elimination is the power method: it is based on the idea that the equilibrium vector is what we expect in the long-term. We can approximate this "long-term" behaviour by simulating a few transitions. For instance, in our example, if we start with the state $[\,1/4 \;\; 1/4 \;\; 1/4 \;\; 1/4\,]^T$, which corresponds to equal chances of being on each webpage, then the next state (that is, after one random click) is

$$T\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/12 \\ 1/3 \\ 5/24 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.083 \\ 0.333 \\ 0.208 \end{bmatrix}.$$

Note that the ranking of the webpages is already A, C, D, B if we stop right here.

The state after that (that is, after two random clicks) is $T^2\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.125 \\ 0.333 \\ 0.167 \end{bmatrix}$, and $T^3\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.396 \\ 0.125 \\ 0.292 \\ 0.188 \end{bmatrix}$.

Observe how we are (overall) approaching the equilibrium vector $\begin{bmatrix} 0.375 \\ 0.125 \\ 0.313 \\ 0.188 \end{bmatrix}$.

Iterating like this is guaranteed to converge to a 1-eigenvector under mild technical assumptions on the transition matrix (for instance, that all its entries be positive; in that case, the other eigenvalues λ satisfy $|\lambda| < 1$ so that their contributions go to zero exponentially, as in Example 93).

Application: Fibonacci numbers

The numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

How fast are they growing? Have a look at ratios of Fibonacci numbers:

$\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.667$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} \approx 1.615$, $\frac{34}{21} \approx 1.619$, ...

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

In other words, it appears that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 100 below.

The crucial insight is the following simple observation:

$$F_{n+2} = F_{n+1} + F_n \quad \text{is equivalent to} \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

In particular, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

Comment. Recurrence equations are discrete analogs of differential equations. We will later see the same idea applied when we reduce the order of a differential equation by introducing additional equations.

Everything we observe here about Fibonacci numbers holds for other sequences that satisfy similar recursion equations.

Theorem 100. (Binet's formula) $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof.

- We already observed that thee recurrence $F_{n+2} = F_{n+1} + F_n$ translates into $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $T = PDP^{-1}$ with

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

  Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = T^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} &= PD^nP^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix} \end{aligned}$$

- Hence, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, which is the claimed formula.

□

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

Comment. It follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Note that, given any Fibonacci-like recursion, we can apply our linear algebra skills in the same fashion. The next example illustrates how this is set up.

Example 101. Suppose the sequence a_n satisfies $a_{n+3} = 3a_{n+2} - 2a_{n+1} + 7a_n$. Write down a matrix-vector version of this recursion.

Solution. $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$

Example 102. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 3a_n$ and $a_0 = -1$, $a_1 = 5$.

(a) Determine the first few terms of the sequence.

(b) Write down a matrix-vector version of the recursion.

(c) Find a Binet-like formula for a_n.

(d) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $-1, 5, 7, 29, 79, 245, 727, 2189, 6559, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

(c) **(solution using matrix powers)** Thus, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.

After some work (do it!), we diagonalize $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$ with $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = PD^nP^{-1}\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix}}_{=\begin{bmatrix} 3^{n+1} & (-1)^{n+1} \\ 3^n & (-1)^n \end{bmatrix}} \underbrace{\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{=\begin{bmatrix} 1 \\ -2 \end{bmatrix}} = \begin{bmatrix} 3^{n+1} - 2(-1)^{n+1} \\ 3^n - 2(-1)^n \end{bmatrix}$.

In particular, $a_n = 3^n - 2(-1)^n$.

(simplified solution) The eigenvalues of $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ are 3 and -1.

Looking back at our work above, we can see that a_n therefore must have a formula of the form $a_n = C_1 \cdot 3^n + C_2 \cdot (-1)^n$ for some unknown constants C_1, C_2 which we still need to figure out

Using the two initial conditions, we get two equations:

(a_0=) $C_1 + C_2 = -1$, ($a_1 =$) $3C_1 - C_2 = 5$.

Solving, we find $C_1 = 1$ and $C_2 = -2$ so that, in conclusion, $a_n = 3^n - 2(-1)^n$.

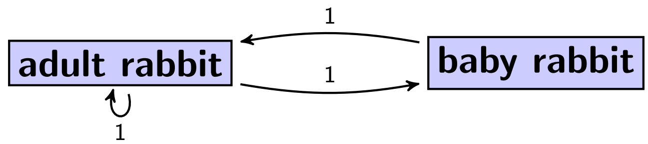
(d) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_1 = 0$.

Review. Fibonacci numbers, Binet formula

Example 103. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let a_t be the number of adult rabbit pairs after t months. Likewise, b_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} a_t + b_t \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix}.$$

That's precisely the transition for the Fibonacci numbers!

It follows that Fibonacci numbers count the number of rabbits in this model.

Comment. Note that the setup is very much as for Markov chains. Here, however, the outgoing values do not add to 100% for each state. Consequently, we cannot expect an equilibrium (and, indeed, the number of rabbits increases without bound).

Definition 104. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = r_1 a_{n+d-1} + r_2 a_{n+d-2} + \ldots + r_d a_n$$

is called ***C*-finite** (or, **constant recursive**) of order d.

For instance. For the Fibonacci numbers, $d = 2$ and $r_1 = r_2 = 1$.

In matrix-vector form. $$\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 & r_2 & \cdots & r_{d-1} & r_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{T} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$$

By the same reasoning as for Fibonacci numbers, C-finite sequences have a Binet-like formula:

Theorem 105. (generalized Binet formula) Suppose the recursion matrix T has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + \ldots + C_d \lambda_d^n$$

for certain numbers $C_1, \ldots, C_d$.

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, and $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$.

Comment. A little more care is needed in the case that eigenvalues are repeated.

Corollary 106. Under the assumptions of the previous theorem, if λ_1 is the eigenvalue with the largest absolute value and $\lambda_1 > 0$, as well as $\alpha_1 \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1$.

Proof. This follows from $a_n = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n$ because, for large n, the term $C_1\lambda_1$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1} + \ldots + C_d\lambda_d^{n+1}}{C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n} = \frac{C_1\lambda_1 + C_2\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\lambda_d\left(\frac{\lambda_d}{\lambda_1}\right)^n}{C_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\left(\frac{\lambda_d}{\lambda_1}\right)^n} \xrightarrow{n\to\infty} \frac{C_1\lambda_1}{C_1} = \lambda_1.$$

□

Example 107. Consider the sequence a_n defined by $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ and $a_0 = 0$, $a_1 = -2$, $a_2 = 2$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, -2, 2, 10, 50, 178, 602, 1930, 6050, \ldots$

Note that this sequence is C-finite of order 3.

(b) The recursion can be translated to $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

Expanding by the 2nd row: $\begin{vmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -6 \\ 1 & -\lambda \end{vmatrix} - \lambda \cdot \begin{vmatrix} 4-\lambda & -6 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6$

The eigenvalues of the transition matrix are the roots of this polynomial: $\lambda = -1, 2, 3$

[You will not be asked to find roots of cubic polynomials by hand.]

Hence, $a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot 3^n$ and we only need to figure out the two unknowns C_1, C_2, C_3.

Using the three initial conditions, we get three equations:

($a_0 =$) $C_1 + C_2 + C_3 = 0$, ($a_1 =$) $-C_1 + 2C_2 + 3C_3 = -2$, ($a_2 =$) $C_1 + 4C_2 + 9C_3 = 2$.

Solving, we find $C_1 = 1$, $C_2 = -2$ and $C_3 = 1$ so that, in conclusion, $a_n = (-1)^n - 2 \cdot 2^n + 3^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$ (the eigenvalue of largest absolute value).

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_3 = 0$.

Example 108. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{5}$. Hence, $a_n = C_1(1+\sqrt{5})^n + C_2(1-\sqrt{5})^n$ for certain numbers C_1, C_2.

[Note that we cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{5})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1}F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Example 109. Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 5a_n$ and $a_0 = 0$, $a_1 = 1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, 1, 2, 9, 28, 101, 342, 1189, 4088, \ldots$

(b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{6}$.

Hence, $a_n = C_1(1+\sqrt{6})^n + C_2(1-\sqrt{6})^n$ and we only need to figure out the values of C_1 and C_2.
Using the two initial conditions, we get two equations:
$(a_0 =)\ C_1 + C_2 = 0$, $(a_1 =)\ C_1(1+\sqrt{6}) + C_2(1-\sqrt{6}) = 1$.
Solving, we find $C_1 = \frac{1}{2\sqrt{6}}$ and $C_2 = -\frac{1}{2\sqrt{6}}$ so that, in conclusion, $a_n = \frac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$.

Comment. Alternatively, we could have proceeded as we did previously in the case of the Fibonacci numbers: starting with the recursion matrix T, we compute its diagonalization $T = PDP^{-1}$. Multiplying out $PD^nP^{-1}\begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$, we obtain the Binet-like formula for a_n. However, this is more work than what we did.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{6} \approx 3.44949$.

Comment. Actually, we don't need the Binet-like formula for this conclusion. Just the eigenvalues and the observation that C_1 cannot be 0 are enough. [We cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{6})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Another brief look at projections (and reflections)

(projections) Suppose that M is the projection matrix for projecting onto a subspace W.

- The 1-eigenspace of M is W.
- The 0-eigenspace of M is $W^\perp$.

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get projected to themselves. But those are precisely the vectors in W (recall that projecting a vector $\boldsymbol{v}$ onto W means producing the vector in W that is closest to $\boldsymbol{v}$). Can you likewise spell out the situation for the 0-eigenspace?

Note that M cannot have further eigenvalues (because the dimensions of W and $W^\perp$ already add up to the dimension of the space that we are working in).

Because the eigenvalues of M are real and the eigenspaces are orthogonal, the matrix M has a diagonalization of the form $M = PDP^T$ (make sure you can explain why!) which implies that M is symmetric (that's because $M^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = M$).

Example 110. Let A be the matrix for orthogonally projecting onto $W = \mathrm{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^{-1}$.

(b) Diagonalize A as $A = PDP^T$.

Comment. This gives us yet another way to compute projection matrices: we can directly write down the matrices P, D for the diagonalization $A = PDP^T$. The main point here is that the diagonalization of a A nicely reveals all the information about the projection.

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional).

We already have a basis for W. On the other hand, $W^\perp = \mathrm{null}\left(\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4 & 0 & -1/4 \\ 0 & 2 & -1/2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

Applying Gram–Schmidt to the basis $\boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (of the 1-eigenspace), we construct the orthogonal basis $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{17} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{17} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}$.

Next, we normalize the vectors $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{17} \begin{bmatrix} -4 \\ 34 \\ 16 \end{bmatrix}, \begin{bmatrix} -1/4 \\ -1/2 \\ 1 \end{bmatrix}$ to $\frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{357}} \begin{bmatrix} -2 \\ 17 \\ 8 \end{bmatrix}, \frac{1}{\sqrt{21}} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 4/\sqrt{17} & -2/\sqrt{357} & -1/\sqrt{21} \\ 0 & 17/\sqrt{357} & -2/\sqrt{21} \\ 1/\sqrt{17} & 8/\sqrt{357} & 4/\sqrt{21} \end{bmatrix}$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{21} \begin{bmatrix} 20 & -2 & 4 \\ -2 & 17 & 8 \\ 4 & 8 & 5 \end{bmatrix}$ as in Example 54.

Example 111. Let A be the matrix for orthogonally projecting onto $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, 0$. The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional). Note that we are lucky and already have an orthogonal basis for W. On the other hand, $W^\perp = \mathrm{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is not invertible (because 0 is an eigenvalue) and therefore also cannot be orthogonal. Like any projection matrix, A is symmetric.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

(reflections) Suppose that M is the matrix for reflecting through the plane W in 3-space.

- The 1-eigenspace of M is W. (dimension 2)
- The -1-eigenspace of M is $W^\perp$. (dimension 1)

In particular, M is symmetric.

Why? By definition, the 1-eigenspace of M consists of those vectors that get reflected to themselves. But those are precisely the vectors in the plane W (only vectors on the plane are unchanged by the reflection). On the other hand, the -1-eigenspace consists of those vectors $\boldsymbol{v}$ that get reflected to $-\boldsymbol{v}$ (the exact opposite direction). These are precisely the vectors orthogonal to the plane.

As in the case of projection matrices, because the eigenvalues are real and the eigenspaces are orthogonal, the reflection matrices are symmetric.

Comment. In this context, the line $W^\perp$ is often called the **normal line** of the plane W.

Example 112. Let A be the matrix for reflecting through the plane $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, -1$. The 1-eigenspace of A is W, and the -1-eigenspace is $W^\perp$.

In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

As in the previous example, $W^\perp = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is invertible (because 0 is not an eigenvalue).

Like any reflection matrix, A is symmetric.

Finally, note that $A^2 = I$ (reflecting twice isn't doing anything), so that $A^{-1} = A$. It follows that A is orthogonal, because $A^{-1} = A = A^T$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. Similarly, a $n \times n$ matrix corresponds to a reflection (through a hyperplane) if and only if it has a $(n-1)$-dimensional 1-eigenspace and a 1-dimensional -1-eigenspace and these two spaces are orthogonal.

An alternative way of computing reflection matrices. Realize that, if $\boldsymbol{n}$ is the vector orthogonal to the plane (i.e. $\boldsymbol{n}$ is the normal vector of the plane), then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

We already observed that $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hence, the reflection of $\boldsymbol{v}$ is $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n}) = \boldsymbol{v} - 2\boldsymbol{n}\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{\boldsymbol{n}\cdot\boldsymbol{n}} = \boldsymbol{v} - 2\frac{\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{v}}{\boldsymbol{n}^T\boldsymbol{n}} = \left(I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}}\right)\boldsymbol{v}$.

Accordingly, the reflection matrix is $A = I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. In other words, we got A from subtracting 2 times the projection matrix onto $\boldsymbol{n}$ from I.

Application: Linear differential equations

Example 113. (warmup) Solve the **differential equation** (DE) $y' = 2$.

Solution. From calculus, we know that the solutions are of the form $y(t) = 2t + C$.

Comment. To get a unique solution, we need to specify additional information, like an initial condition.

Example 114. (warmup) Solve the **initial value problem** (IVP) $y' = 2$, $y(0) = 1$.

Solution. This has the unique solution $y(t) = 2t + 1$.

Example 115. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Solution. $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

(exponential function) e^t is the unique solution to $y' = y$, $y(0) = 1$.

From here, it follows that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$.

The latter is the Taylor series for e^t at $t = 0$ that we have seen in Calculus II.

Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.

Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{t^3}{3!} = \frac{t^2}{2!}$.

Example 116. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Solution. Indeed, if $y(t) = Ce^{3t}$, then $y'(t) = 3Ce^{3t} = 3y(t)$.

Comment. It is important to realize that we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

Example 117. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Solution. As in the previous example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$.

Since $y(0) = Ce^0 = C = y_0$, we conclude that the unique solution to the IVP is $y(t) = e^{at}y_0$.

Comment. It looks silly to write $e^{at}y_0$ instead of y_0e^{at} here, but we will soon replace the number a with a matrix A, and in that case only $e^{At}y_0$ makes sense.

Example 118. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrr} y_1' &=& 2y_1 & & \\ y_2' &=& -y_1 & +3y_2 & +y_3 \\ y_3' &=& -y_1 & +y_2 & +3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form, this becomes

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The key idea will be to solve $\boldsymbol{y}' = A\boldsymbol{y}$ by introducing e^{At}.

Theorem 119. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Recall from Example 117 that the solution to $y' = ay$, $y(0) = y_0$ is $y(t) = e^{at}y_0$. Here, however, At is a matrix and so we need to make sense of the matrix exponential. Next time, we will define e^A by the familiar Taylor series for e^x.

Definition 120. Let A be $n\times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Why? As a consequence of this definition (which is the motivation for that definition in the first place),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{At} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right]\\ &= 0 + A + A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}.\end{aligned}$$

Therefore, $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$ indeed solves the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

How to actually compute e^A? Well, this Taylor series involves the powers A^n of A. How would you compute, say, A^{100}? The answer is diagonalization!

Theorem 121. Suppose $A = PDP^{-1}$. Then, $e^A = Pe^DP^{-1}$.

Why? Recall that, if $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$.

$$\begin{aligned}e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots\\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)P^{-1} = Pe^DP^{-1}\end{aligned}$$

Comment. By the same argument, if $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ for every "nice" function f.
Here, "nice" means that f has a convergent Taylor series $f(x) = \sum_{n\geqslant 0} a_n x^n$.
More explicitly, if $A = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,P^{-1}$, then $f(A) = P\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))\,P^{-1}$.

Example 122. If $A = \begin{bmatrix} 2 & 0\\ 0 & 5 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0\\ 0 & 5^{100} \end{bmatrix}$.

Example 123. If $A = \begin{bmatrix} 2 & 0\\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0\\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0\\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0\\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for every diagonal matrix D.

In particular, for $At = \begin{bmatrix} 2t & 0\\ 0 & 5t \end{bmatrix}$, $e^{At} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0\\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0\\ 0 & (5t)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2t} & 0\\ 0 & e^{5t} \end{bmatrix}$.

Example 124. (homework) Diagonalize $A = \begin{bmatrix} 2 & 0 & 0\\ -1 & 3 & 1\\ -1 & 1 & 3 \end{bmatrix}$.

Solution. (final solution only) $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$.

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Example 125. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, we diagonalize:

  For $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$. (That's homework!)

- We can then compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix} \end{aligned}$$

Comment. It is not necessary to compute $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$ (of course, you could do it, but that's more work).
Instead, recall that $A^{-1}\boldsymbol{b}$ is the unique solution to $A\boldsymbol{x} = \boldsymbol{b}$. Here, solving $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, we find $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Check. $\boldsymbol{y} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$ indeed solves the original problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} + 4e^{4t} \\ 4e^{4t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 + 1 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example 126. We only discuss linear differential equations (DEs). Non-linear DEs include $y' = y^2 + 1$ or the second-order equation $y'' = \sin(ty') + y$.

The order of a DE indicates the highest occuring derivative.

Note, however, that $y'' = \sin(t)y' + y$ is a linear DE, because y and its derivatives occur linearly.

We will see here how to solve those linear DEs which have constant coefficients. That is, the coefficients of y are constants, as opposed to functions (like $\sin(t)$) depending on t.

Review.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

  Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

- If we have the diagonalization $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$ (and $e^{At} = Pe^{Dt}P^{-1}$).

- If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Example 127. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$ has characteristic polynomial $-\lambda(1-\lambda) - 2 = (\lambda+1)(\lambda-2)$.

  Hence, the eigenvalues of A are $-1, 2$.

  The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

  The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{2t} \end{bmatrix}}_{\begin{bmatrix} 2e^{-t} & -e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix}} \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2e^{-t} + e^{2t} \\ e^{-t} - e^{2t} \end{bmatrix} \end{aligned}$$

Example 128. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Note. The "trick" of looking at the pair $\begin{bmatrix} y \\ y' \end{bmatrix}$ instead of a single function is what we used to translate the Fibonacci recurrence into a 2×2 system.

Example 129. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

The Jordan normal form

Note that we currently only know how to compute e^{At} when A is diagonalizable. Our next goal is to be able to compute the matrix exponential for all matrices.

Example 130. Diagonalize, if possible, the matrix $A = \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}$.

Solution. The eigenvalues of A are $4, 4$.

However, the 4-eigenspace $\text{null}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1-dimensional.

Hence, A is not diagonalizable.

Definition 131. A **$\boldsymbol{\lambda}$-Jordan block** is a matrix of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$.

Note that if this matrix is $m \times m$, then its only eigenvalue is λ (repeated m times).

As in the previous example, the λ-eigenspace is 1-dimensional (which is as small as possible).

Theorem 132. (Jordan normal form) Every $n \times n$ matrix A can be written as $A = PJP^{-1}$, where J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}$$

with each J_i a Jordan block. J is called the **Jordan normal form** of A.

Up to the ordering of the Jordan blocks, the Jordan normal form of A is unique.

Comment. If A is diagonalizable, then J is just a usual diagonal matrix.

Example 133. What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues 4, $4, 4$?

Solution. $\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$

The dimension of the 4-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 4 & & \\ & 4 & 1 \\ & & 4 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you know from diagonalization).

Example 134. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Review. Jordan normal form

Example 135.

(a) What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$?

(b) What are the possible Jordan normal forms of a 4×4 matrix with eigenvalues $3, 3, 3, 3$?

(c) What if the matrix is 5×5 and has eigenvalues $4, 4, 3, 3, 3$?

Solution.

(a) $\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

Comment. Note that, say, $\begin{bmatrix} 3 & 1 & \\ & 3 & \\ & & 3 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$ because the ordering of the diagonal blocks does not matter (as you known from diagonalization).

(b) Now, there are 5 possibilities:

$\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 4, 3, 2, 2, 1, respectively.

(c) $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$

Note that this is just all possible (namely, 3) Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$ combined with all possible (namely, 2) Jordan normal forms of a 2×2 matrix with eigenvalues $4, 4$. In total, that makes $3 \cdot 2 = 6$ possibilities.

Comment. Let $p(n)$ be the number of inequivalent Jordan normal forms of an $n \times n$ matrix with a single eigenvalue, n times repeated. We have seen that $p(2) = 2$, $p(3) = 3$, $p(4) = 5$. Note that $p(n)$ is equal to the number of ways of writing n as an ordered sum of positive integers: for instance, $p(4) = 5$ because $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$.

$p(n)$ is referred to as the **partition function** and, surprisingly, is a remarkably interesting mathematical object.

https://en.wikipedia.org/wiki/Partition_function_(number_theory)

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Example 136. (summary of small cases)

(a) There are 2 possible Jordan normal forms of a 2×2 matrix with eigenvalues λ,λ.

Namely. $\begin{bmatrix}\lambda & \\ & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & 1\\ & \lambda\end{bmatrix}$

(b) There are 3 possible Jordan normal forms of a 3×3 matrix with eigenvalues λ,λ,λ.

Namely. $\begin{bmatrix}\lambda & & \\ & \lambda & \\ & & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & & \\ & \lambda & 1\\ & & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{bmatrix}$

(c) There are 5 possible Jordan normal forms of a 4×4 matrix with eigenvalues $\lambda,\lambda,\lambda,\lambda$.

Namely. $\begin{bmatrix}\lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & & & \\ & \lambda & & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & & & \\ & \lambda & 1 & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix}$, $\begin{bmatrix}\lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1\\ & & & \lambda\end{bmatrix}$

Example 137. What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues $3, 3, 7, 7, 7, 7$?

Solution. There are $2\cdot 5 = 10$ possible Jordan normal forms for such a matrix:

$$\begin{bmatrix}3&&&&&\\&3&&&&\\&&7&&&\\&&&7&&\\&&&&7&\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&&&&&\\&3&&&&\\&&7&1&&\\&&&7&&\\&&&&7&\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&&&&&\\&3&&&&\\&&7&1&&\\&&&7&&\\&&&&7&1\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&&&&&\\&3&&&&\\&&7&1&&\\&&&7&1&\\&&&&7&\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&&&&&\\&3&&&&\\&&7&1&&\\&&&7&1&\\&&&&7&1\\&&&&&7\end{bmatrix},$$

$$\begin{bmatrix}3&1&&&&\\&3&&&&\\&&7&&&\\&&&7&&\\&&&&7&\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&1&&&&\\&3&&&&\\&&7&1&&\\&&&7&&\\&&&&7&\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&1&&&&\\&3&&&&\\&&7&1&&\\&&&7&&\\&&&&7&1\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&1&&&&\\&3&&&&\\&&7&1&&\\&&&7&1&\\&&&&7&\\&&&&&7\end{bmatrix}, \begin{bmatrix}3&1&&&&\\&3&&&&\\&&7&1&&\\&&&7&1&\\&&&&7&1\\&&&&&7\end{bmatrix}$$

Example 138. How many different Jordan normal forms are there in the following cases?

(a) A 8×8 matrix with eigenvalues $1,1,2,2,2,4,4,4$?

(b) A 11×11 matrix with eigenvalues $1,1,1,2,2,2,2,4,4,4,4$?

Solution.

(a) $2\cdot 3\cdot 3 = 18$ possible Jordan normal forms

(b) $3\cdot 5\cdot 5 = 75$ possible Jordan normal forms

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Review.

- Let A be $n \times n$. The matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Then, $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$.

Why? $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$

- If $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

Example 139. The matrix exponential shares many other properties of the usual exponential:

- $e^Ae^B = e^{A+B} = e^Be^A$ if $AB = BA$

Why the condition $\boldsymbol{AB = BA}$? By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$ In order to simplify that to

$$e^Ae^B = \left(I + A + \frac{A^2}{2!} + \ldots\right)\left(I + B + \frac{B^2}{2!} + \ldots\right),$$

we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- e^A is invertible and $(e^A)^{-1} = e^{-A}$

Why? That actually follows from the previous property.

Example 140. Compute e^{At} for $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$.

Solution.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$. Note that $2I$ and N commute.

  Hence, $e^{At} = e^{2It + Nt} = e^{2It}e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!}N^2 + \ldots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It}e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ & e^{2t} \end{bmatrix}$.

Advanced. Can you show that $A^n = \begin{bmatrix} 2^n & n2^{n-1} \\ & 2^n \end{bmatrix}$?

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Example 141. Solve the differential equation

$$\boldsymbol{y}' = \underbrace{\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}}_{A} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_0}$$

Solution. Repeating the work in the previous example, the solution to the differential equation is

$$\begin{aligned}
\boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 \\
&= e^{2It+Nt}\boldsymbol{y}_0 \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= e^{2It}e^{Nt}\boldsymbol{y}_0 \qquad \text{(because } 2It \text{ and } Nt \text{ commute)} \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\left(1 + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots\right)\boldsymbol{y}_0 \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}(1 + Nt)\boldsymbol{y}_0 \qquad \text{(because } N^2 = \mathbf{0}) \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix}.
\end{aligned}$$

Check. We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$.
Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

Comment. For applications, having solutions like $te^{\lambda t}$ or $t\cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.

Important comment. Note that we can immediately see from the solution that the original matrix A is not diagonalizable: there is a term te^{2t}, whereas in the diagonalizable case we would only see exponentials like e^{2t} by themselves.

In our upcoming discussion of complex numbers we will see that e^{2it} (here, $2i$ would be the eigenvalue) can be rewritten in terms of $\cos(2t)$ and $\sin(2t)$. Both of these are periodic and bounded, so that the same is true for every linear combination.

In that case, if the eigenvalue $2i$ was repeated in such a way that the matrix A is not diagonalizable, then we would get the functions $t\cos(2t)$ and $t\sin(2t)$ in our solutions. These, however, are not bounded! This phenomenon (getting solutions that are unbounded under the right/wrong circumstances) is called **resonance**.

https://en.wikipedia.org/wiki/Resonance

Understanding when resonance occurs is of crucial importance for practical applications.

Example 142. Solve the IVP $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- We first diagonalize $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
  - $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$, so the eigenvalues are ± 1.
  - The 1-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}}_{=\begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}} \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix} \end{aligned}$$

Check. Indeed, $y_1 = \frac{1}{2}(e^t + e^{-t})$ and $y_2 = \frac{1}{2}(e^t - e^{-t})$ satisfy the system of differential equations $y_1' = y_2$ and $y_2' = y_1$ as well as the initial conditions $y_1(0) = 1$, $y_2(0) = 0$.

Comment. You have actually met these functions in Calculus! $y_1 = \cosh(t)$ and $y_2 = \sinh(t)$. Check out the next example for the connection to $\cos(t)$ and $\sin(t)$.

Example 143.

(a) Solve the IVP $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Show that $\boldsymbol{y} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ solves the same IVP. What do you conclude?

Solution.

(a) $A = PDP^{-1}$ with $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

The system is therefore solved by:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} \\ -e^{-it} \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} ie^{it} + ie^{-it} \\ e^{it} - e^{-it} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix} \end{aligned}$$

(b) Clearly, $\boldsymbol{y}(0) = \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. On the other hand, $y_1' = -\sin(t) = -y_2$ and $y_2' = \cos(t) = y_1$, so that $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$. Since the solution to the IVP is unique, it follows that $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix}$.

We have just discovered **Euler's identity**!

Theorem 144. (Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Another short proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Rotation matrices

Example 145. Write down a 2×2 matrix Q for rotation by angle θ in the plane.

Comment. Why should we even be able to represent something like rotation by a matrix? Meaning that $Q\boldsymbol{x}$ should be the vector $\boldsymbol{x}$ rotated by θ. Recall from Linear Algebra I that every **linear map** can be represented by a matrix. Then think about why rotation is a linear map.

Solution. We can determine Q by figuring out $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the first column of Q) and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the second column of Q).

Since $Q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, we conclude that $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Comment. Note that we don't need previous knowledge of $\cos$ and $\sin$. We could have introduced these trig functions on the spot.

Comment. Note that it is geometrically obvious that Q is orthogonal. (Why?)

It is clear that $\left\|\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\|^2 = 1$. Noting that $\left\|\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right\|^2 = \cos^2\theta + \sin^2\theta$, we have rediscovered Pythagoras.

Advanced comment. Actually, every orthogonal 2×2 matrix Q with $\det(Q) = 1$ is a rotation by some angle θ. Orthogonal matrices with $\det(Q) = -1$ are reflections.

Example 146. As in the previous example, let Q_θ be the 2×2 matrix for rotation by angle θ in the plane. What is $Q_\alpha Q_\beta$?

Solution. Note that $Q_\alpha Q_\beta \boldsymbol{x}$ first rotates $\boldsymbol{x}$ by angle β and then by angle α. For geometric reasons, it is obvious that this is the same as if we rotated $\boldsymbol{x}$ by $\alpha + \beta$. It follows that $Q_\alpha Q_\beta = Q_{\alpha+\beta}$.

Comment. This allows us to derive interesting trig identities:

$$\begin{aligned} Q_\alpha Q_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \ldots \\ \ldots & \ldots \end{bmatrix} \\ Q_{\alpha+\beta} &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \end{aligned}$$

It follows that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Comment. If we set $\beta = \alpha$, this simplifies to $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$, the double angle formula that you have probably used countless times in Calculus.

Comment. Similarly, we find an identity for $\sin(\alpha+\beta)$. Spell it out!

More on complex numbers

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at Example 82.

- The **absolute value** of the complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$.
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$.
  Note that $\|\boldsymbol{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\boldsymbol{z}}^T \boldsymbol{z}$.

Definition 147.

- For every matrix A, its **conjugate transpose** is $A^* = (\bar{A})^T$.
- The **dot product** (inner product) of complex vectors is $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^* \boldsymbol{w}$.
- A complex $n \times n$ matrix A is **unitary** if $A^* A = I$.

Comment. A^* is also written A^H (or $A^\dagger$ in quantum mechanics) and called the Hermitian conjugate.

Comment. For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

Comment. Unitary matrices are the complex version of orthogonal matrices. (A real matrix if unitary if and only if it is orthogonal.)

Example 148. What is the norm of the vector $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\|^2 = \begin{bmatrix} 1+i & 2-3i \end{bmatrix} \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} = |1-i|^2 + |2+3i|^2 = 2 + 13$. Hence, $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\| = \sqrt{15}$.

Example 149. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

Solution. $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

Example 150. What is $\frac{1}{2+3i}$?

Solution. $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$.

In general. $\frac{1}{z} = \frac{\bar{z}}{z\,\bar{z}} = \frac{\bar{z}}{|z|^2}$

Example 151. (extra) We can identify complex numbers $x + iy$ with vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$. Then, what is the geometric effect of multiplying with i?

Solution. Algebraically, the effect of multiplying $x + iy$ with i obviously is $i(x + iy) = -y + ix$.

Since multiplication with i is obviously linear, we can represent it using a 2×2 matrix J acting on vectors $\begin{bmatrix} x \\ y \end{bmatrix}$.

$J \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (this is the same as saying $i \cdot 1 = i$) and $J \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (this is the same as saying $i \cdot i = -1$).

Hence, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This is precisely the rotation matrix for a rotation by $90°$.

In other words, multiplication with i has the geometric effect of rotating complex numbers by $90°$.

Comment. The relation $i^2 = -1$ translates to $J^2 = -I$.

Complex numbers as 2×2 matrices. In light of the above, we can express complex numbers $x + iy$ as the 2×2 matrix $xI + yJ = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. Adding and multiplying these matrices behaves exactly the same way as adding or multiplying the complex numbers directly.

For instance, $(2+3i)(4-i) = 8 + 10i - 3i^2 = 11 + 10i$ versus $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ 10 & 11 \end{bmatrix}$.

Likewise for inverses: $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$ versus $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

Extra: More details on the spectral theorem

Let us add $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ to our notations for the dot product: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^T \boldsymbol{w} = \boldsymbol{v} \cdot \boldsymbol{w}$.

- In our story of orthogonality, the important player has been the dot product. However, one could argue that the fundamental quantity is actually the norm:
  $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \frac{1}{4}(\|\boldsymbol{v} + \boldsymbol{w}\|^2 - \|\boldsymbol{v} - \boldsymbol{w}\|^2)$. See Example 24.

- Accepting the dot product as immensely important, we see that symmetric matrices (i.e. matrices A such that $A = A^T$) are of interest.
  For every matrix A, $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A^T \boldsymbol{w} \rangle$.
  It follows that, a matrix A is symmetric if and only if $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

- Similarly, let Q be an orthogonal matrix (i.e. Q is a square matrix with $Q^T Q = I$).
  Then, $\langle Q\boldsymbol{v}, Q\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$.
  In fact, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.
  Comment. We observed in Example 145 that orthogonal matrices Q correspond to rotations ($\det Q = 1$) or reflections ($\det Q = -1$) [or products thereof]. The equality $\langle Q\boldsymbol{v}, Q\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ encodes the fact that these types (and only these!) of geometric transformations preserve angles and lengths.

(Spectral theorem)

A $n \times n$ matrix A is symmetric if and only if it can be decomposed as $A = PDP^T$, where

- D is a diagonal matrix, $(n \times n)$

The diagonal entries λ_i are the **eigenvalues** of A.

- P is orthogonal. $(n \times n)$

The columns of P are **eigenvectors** of A.

Note that, in particular, A is always diagonalizable, the eigenvalues (and hence, the eigenvectors) are all real, and, most importantly, the eigenspaces of A are orthogonal.
The "only if" part says that, if A is symmetric, then we get a diagonalization $A = PDP^T$. The "if" part says that, if $A = PDP^T$, then A is symmetric (which follows from $A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$).

Let us prove the following important parts of the spectral theorem.

Theorem 152.

(a) If A is symmetric, then the eigenspaces of A are orthogonal.

(b) If A is real and symmetric, then the eigenvalues of A are real.

Proof.

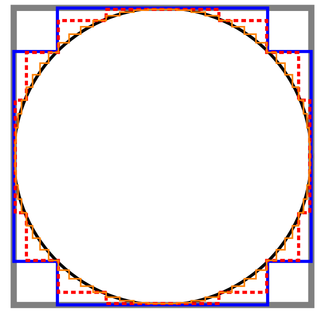
(a) We need to show that, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{0}$.
Suppose that $A\boldsymbol{v} = \lambda \boldsymbol{v}$ and $A\boldsymbol{w} = \mu \boldsymbol{w}$ with $\lambda \neq \mu$.
Then, $\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \lambda \boldsymbol{v}, \boldsymbol{w} \rangle = \langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A^T \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \mu \boldsymbol{w} \rangle = \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$.
However, since $\lambda \neq \mu$, $\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ is only possible if $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

(b) Suppose λ is a nonreal eigenvalue with nonzero eigenvector $\boldsymbol{v}$. Then, $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector and, since $\lambda \neq \bar{\lambda}$, we have two eigenvectors with different eigenvalues. By the first part, these two eigenvectors must be orthogonal in the sense that $\bar{\boldsymbol{v}}^T \boldsymbol{v} = 0$. But $\bar{\boldsymbol{v}}^T \boldsymbol{v} = \boldsymbol{v}^* \boldsymbol{v} = \|\boldsymbol{v}\|^2 \neq 0$. This shows that it is impossible to have a nonzero eigenvector for a nonreal eigenvalue. □

Let us highlight the following point we used in our proof:

Let A be a real matrix. If $\boldsymbol{v}$ is a λ-eigenvector, then $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector.

See, for instance, Example 82. This is just a consequence of the basic fact that we cannot algebraically distinguish between $+i$ and $-i$.



Remark 153. (April Fools' Day!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.

Can you pin-point the fallacy in this argument?

Comment. We'll actually come back to this. It's related to linear algebra in infinite dimensions.

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