Review: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \Longleftrightarrow\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \Longleftrightarrow\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

More precisely, we find a basis of eigenvectors for the λ-**eigenspace** $\operatorname{null}(A - \lambda I)$.

Example 16. $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ has one eigenvector that is "easy" to see. Do you see it?

Solution. Note that $A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector.

Just for contrast. Note that $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$, where

- the columns of P are the eigenvectors, and
- the diagonal matrix D has the eigenvalues on the diagonal.

Such a diagonalization is possible if and only if A has enough (independent) eigenvectors.

Comment. If you don't quite recall why these choices result in the diagonalization $A = PDP^{-1}$, note that the diagonalization is equivalent to $AP = PD$.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix}$$
$$= \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

Example 17. Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues and bases for the eigenspaces of A.

(b) Diagonalize A. That is, determine matrices P and D such that $A = PDP^{-1}$.

Solution.

(a) By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda) - 2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 5$.

Comment. At this point, we know that we will find one eigenvector for $\lambda = 5$ (more precisely, the 5-eigenspace definitely has dimension 1). On the other hand, the 2-eigenspace might have dimension 2 or 1. In order for A to be diagonalizable, the 2-eigenspace must have dimension 2. (Why?!)

- The 5-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right)$. Proceeding as in Example 14, we obtain

  $$\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}.$$

  In other words, the 5-eigenspace has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

- The 2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$. Proceeding as in Example 15, we obtain

  $$\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

  In other words, the 2-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

  Comment. So, indeed, the 2-eigenspace has dimension 2. In particular, A is diagonalizable.

(b) A possible choice is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Comment. However, many other choices are possible and correct. For instance, the order of the eigenvalues in D doesn't matter (as long as the same order is used for P). Also, for P, the columns can be chosen to be any other set of eigenvectors.

Example 18. (extra practice) Diagonalize, if possible, the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution. For instance, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & & \\ & 2 & \\ & & 2 \end{bmatrix}$. B is not diagonalizable.

For instance, $C = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$.

Example 19. (review) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Recall that $(AB)^T = B^T A^T$. This reflects the fact that, in the column-centric versus the row-centric interpretation of matrix multiplication, the order of the matrices is reversed.

Comment. When working with complex numbers, the fundamental role is not played by the transpose but by the **conjugate transpose** instead (we'll see that in our discussion of orthogonality): $A^* = \overline{A^T}$.

For instance, if $A = \begin{bmatrix} 1-3i & 5i \\ 2+i & 3 \end{bmatrix}$, then $A^* = \begin{bmatrix} 1+3i & 2-i \\ -5i & 3 \end{bmatrix}$.

Orthogonality

The inner product and distances

Definition 20. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \; = \; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 21. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

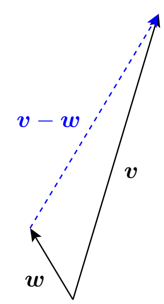
Definition 22.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \; = \; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \; = \; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 23. For instance, in $\mathbb{R}^2$, $\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 24. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

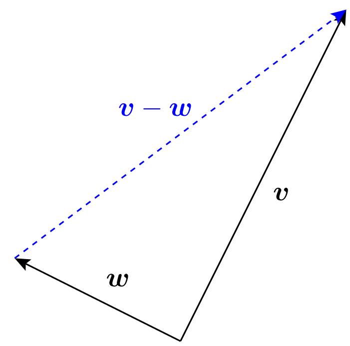
Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

Comment. This is a vector version of $(x - y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

Definition 25. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$\iff \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \underbrace{\|\boldsymbol{v} - \boldsymbol{w}\|^2}_{\substack{= \|\boldsymbol{v}\|^2 - 2\boldsymbol{v}\cdot\boldsymbol{w} + \|\boldsymbol{w}\|^2 \\ \text{(by previous example)}}}$

$\iff -2\boldsymbol{v} \cdot \boldsymbol{w} = 0$

$\iff \boldsymbol{v} \cdot \boldsymbol{w} = 0$

Example 26. Determine a basis for the **orthogonal complement** of (the span of) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

What are we looking for? The orthogonal complement of $\boldsymbol{v}$ consists of all vectors that are orthogonal to $\boldsymbol{v}$. More generally, the orthogonal complement of a space V consists of all vectors that are orthogonal to every vector in V.

Solution. (staring/intution) We are working in 3-dimensional space and already have 1 vector. The vectors orthogonal to it lie in a $3 - 1 = 2$-dimensional space (a plane).

Two of the vectors orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Knowing that the orthogonal complement has dimension 2, we conclude that $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a basis.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

[Note how the dimensions add up to the dimension of the entire space: $1 + 2 = 3$.]

Solution. (professional) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ (dot product!) is the same as $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ (matrix product!).

Hence, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is the same as $\operatorname{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$.

Computing a basis for $\operatorname{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$ is easy since $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ is already in RREF.

Note that the general solution to $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\boldsymbol{x} = 0$ is $\begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $\operatorname{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$ therefore is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. (Check that these are indeed orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$!)

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Example 27. Determine a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$.

Solution. We are looking for vectors $\boldsymbol{x}$ such that $\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ and $\begin{bmatrix}3 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$.

The two equations can be combined into a single one: $\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}$.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$ is the same as $\operatorname{null}\left(\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\end{bmatrix}\right)$.

It remains to compute a basis for that null space:

$$\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\end{bmatrix}\overset{R_2-3R_1\Rightarrow R_2}{\rightsquigarrow}\begin{bmatrix}1 & 2 & 1\\0 & -5 & -1\end{bmatrix}\quad\overset{\text{back-substitution}}{\rightsquigarrow}\quad\begin{bmatrix}-3/5s\\-1/5s\\s\end{bmatrix}$$

Hence, a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$ is $\begin{bmatrix}-3/5\\-1/5\\1\end{bmatrix}$.

Check. $\begin{bmatrix}-3/5\\-1/5\\1\end{bmatrix}$ is indeed orthogonal to both $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\\2\end{bmatrix}$.

Just to make sure. Why was it clear that the orthogonal complement is 1-dimensional?

Important. Note that $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}=\operatorname{col}(A)$ for $A=\begin{bmatrix}1 & 3\\2 & 1\\1 & 2\end{bmatrix}$.

In our computation we showed, and then used, that the orthogonal complement is $\operatorname{null}(A^T)$.

By the same reasoning, this is true for any matrix A: we always have that $\operatorname{col}(A)^\perp=\operatorname{null}(A^T)$. This is a crucial part of the fundamental theorem that we discuss next.

The fundamental theorem

Example 28. The four **fundamental subspaces** associated with a matrix A are

$$\operatorname{col}(A), \quad \operatorname{row}(A), \quad \operatorname{null}(A), \quad \operatorname{null}(A^T).$$

Note that $\operatorname{row}(A) = \operatorname{col}(A^T)$. (In particular, we usually write vectors in $\operatorname{row}(A)$ as column vectors.)

Definition 29. $\operatorname{null}(A^T)$ is the **left null space** of A.

Why that name? Recall that, by definition $\boldsymbol{x}$ is in $\operatorname{null}(A) \iff A\boldsymbol{x} = \mathbf{0}$.
Likewise, $\boldsymbol{x}$ is in $\operatorname{null}(A^T) \iff A^T\boldsymbol{x} = \mathbf{0} \iff \boldsymbol{x}^T A = \mathbf{0}$.

[Recall that $(AB)^T = B^T A^T$. In particular, $(A^T\boldsymbol{x})^T = \boldsymbol{x}^T A$, which is what we used in the last equivalence.]

Theorem 30. **(Fundamental Theorem of Linear Algebra, Part I)**
Let A be an $m \times n$ matrix of **rank** r.

- $\dim \operatorname{col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \operatorname{row}(A) = r$ (subspace of $\mathbb{R}^n$) $\qquad \operatorname{row}(A) = \operatorname{col}(A^T)$
- $\dim \operatorname{null}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \operatorname{null}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Example 31. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Solution. Make sure that, for such a simple matrix, you can see all of these that at a glance!

$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$, $\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$, $\operatorname{null}(A^T) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Important observation. The basis vectors for $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are orthogonal! $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Vectors in $\operatorname{null}(A)$ are orthogonal to vectors in $\operatorname{row}(A)$.
In short, $\operatorname{null}(A)$ is orthogonal to $\operatorname{row}(A)$.

Why? Suppose that $\boldsymbol{x}$ is in $\operatorname{null}(A)$. That is, $A\boldsymbol{x} = \mathbf{0}$. But think about what $A\boldsymbol{x} = \mathbf{0}$ means (row-product rule). It means that the inner product of every row with $\boldsymbol{x}$ is zero. Which implies that $\boldsymbol{x}$ is orthogonal to the row space.

Definition 32. As done in the observation above, we say that two subspaces V and W of $\mathbb{R}^n$ are **orthogonal** if and only if every vector in V is orthogonal to every vector in W.
The **orthogonal complement** of W is the space $W^\perp$ of all vectors that are orthogonal to W.

Exercise. Show that the orthogonal complement is indeed a vector space.

Theorem 33. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\text{null}(A)$ is orthogonal to $\text{row}(A)$. (both subspaces of $\mathbb{R}^n$)

Note that $\dim \text{null}(A) + \dim \text{row}(A) = n$. Hence, the two spaces are orthogonal complements.

- $\text{null}(A^T)$ is orthogonal to $\text{col}(A)$.

Again, the two spaces are orthogonal complements. (This is just the first part with A replaced by A^T.)

Example 34. Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$. Check that $\text{null}(A)$ and $\text{row}(A)$ are orthogonal complements.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & -3 & -9 \end{bmatrix} \overset{R_3 - \frac{3}{2}R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{-\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 - R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$, $\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

$\text{null}(A)$ and $\text{row}(A)$ are indeed orthogonal, as certified by:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$$

In fact, $\text{null}(A)$ and $\text{row}(A)$ are orthogonal complements because the dimensions add up to $2 + 2 = 4$.

In particular, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ form a basis of all of $\mathbb{R}^4$.

Just to make sure. Because $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is orthogonal to both basis vectors, it is orthogonal to every vector in $\text{row}(A)$.

Vectors in $\text{row}(A)$ are of the form $\boldsymbol{v} = a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$. Then, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \boldsymbol{v} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 0.$

Example 35. **(extra)** Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that $\text{null}(A)$ and $\text{row}(A)$ (as well as $\text{null}(A^T)$ and $\text{col}(A)$) are orthogonal complements.

Partial solution. One can almost see that $\text{rank}(A) = 3$. Hence, the dimensions of the fundamental subspaces are ...

Any serious linear algebra problems are done by a machine. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at sagemath.org. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at cocalc.com from any browser. For short computations, like the one below, you can also just use the input field on our course website.

Sage is built as a **Python** library, so any Python code is valid. Here, we will just use it as a fancy calculator.

Let's revisit Example 34 and let Sage do the work for us:

```
Sage] A = matrix([[1,2,1],[2,4,0],[3,6,0]])
```

```
Sage] A.rref()
```

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, if we wanted to compute a basis for $\operatorname{null}(A^T)$, we can simply do:

```
Sage] A.transpose().rref()
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Here are some other standard things we might be interested in (compare with Example 17):

```
Sage] A = matrix([[4,0,2],[2,2,2],[1,0,3]])
```

```
Sage] A.eigenvalues()
```

$$[5, 2, 2]$$

```
Sage] A.eigenvectors_right()
```

$$\left[\left(5, \left[\left(1,\ 1,\ \frac{1}{2}\right)\right], 1\right), (2, [(1,\ 0,\ -1), (0,\ 1,\ 0)], 2)\right]$$

```
Sage] A.eigenmatrix_right()
```

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1}{2} & -1 & 0 \end{bmatrix}\right)$$

```
Sage] A.rank()
```

3

```
Sage] A.determinant()
```

$$20$$

```
Sage] A.inverse()
```

$$\begin{bmatrix} \frac{3}{10} & 0 & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{2} & -\frac{1}{5} \\ -\frac{1}{10} & 0 & \frac{2}{5} \end{bmatrix}$$