Function spaces, II

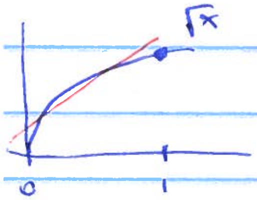
functions $f : [a,b] \to \mathbb{R}$

natural dot product:

$$\langle f, g \rangle := \int_a^b f(t)\, g(t)\, dt$$

projection of f onto g

$$= \frac{\langle f,g \rangle}{\langle g,g \rangle} g$$



EG What is the best approximation of $f(x) = \sqrt{x}$ on $[0,1]$ using a function $y = ax+b$?

$= $ orth. proj. of $\sqrt{x}$ onto $\boxed{\text{span}\{1, x\}} = \text{span}\{1, x-\frac{1}{2}\}$

CAREFUL! $\neq \dfrac{\langle \sqrt{x}, 1 \rangle}{\langle 1, 1 \rangle} 1 + \dfrac{\langle \sqrt{x}, x \rangle}{\langle x, x \rangle} x$

because $1, x$ <u>not</u> orthogonal:

$$\langle 1, x \rangle = \int_0^1 1 \cdot t\, dt = \left[\tfrac{1}{2} t^2\right]_0^1 = \tfrac{1}{2} \neq 0$$

- Gram-Schmidt on $1, x$

$$q_1 = 1 \qquad q_2 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_0^1 t\, dt}{\int_0^1 1\, dt} = x - \frac{1}{2}$$

$$\int_0^1 \sqrt{t}\, dt = \left[\tfrac{2}{3} t^{3/2}\right]_0^1 = \tfrac{2}{3}$$

$\Rightarrow$ best approx

$$= \frac{\langle \sqrt{x}, 1 \rangle}{\underbrace{\langle 1, 1 \rangle}_{=1}} 1 + \frac{\langle \sqrt{x}, x - \frac{1}{2} \rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle} (x - \tfrac{1}{2})$$

$$= \frac{2}{3} + \frac{12}{15}(x - \tfrac{1}{2})$$

$$= \frac{4}{5} x + \frac{4}{15}$$

$$\int_0^1 \sqrt{t}(t - \tfrac{1}{2})\, dt = \int_0^1 \left(t^{3/2} - \tfrac{1}{2} t^{1/2}\right) dt = \left[\tfrac{2}{5} t^{5/2} - \tfrac{1}{3} t^{3/2}\right]_0^1 = \tfrac{2}{5} - \tfrac{1}{3} = \tfrac{1}{15}$$

$$\int_0^1 (t - \tfrac{1}{2})^2\, dt = \int_0^1 (t^2 - t + \tfrac{1}{4})\, dt = \left[\tfrac{1}{3} t^3 - \tfrac{1}{2} t^2 + \tfrac{1}{4} t\right]_0^1 = \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{1}{12}$$