matrix approximation

given: A $m \times n$ $\quad$ $\operatorname{rank}(A) = r$

wanted: B $m \times n$, $\operatorname{rank}(B) = s < r$

best approximation of A

How to find B?

diagonal entries $\sigma_1 > \sigma_2 > \cdots \geq 0$

- $\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V}^T$ $\quad$ SVD

$$\Rightarrow B = \underset{m\times s}{U_s}\,\underset{s\times s}{\Sigma_s}\,\underset{s\times n}{V_s^T}$$

first $\underline{s}$ col's of U, V

$=$ diag $s \times s$ with entries $\sigma_1, \sigma_2, \ldots, \sigma_s$

EG Best rank 1 approx. of $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

earlier: $A = U \Sigma V^T$

$$= \begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^T$$

$$\Rightarrow \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T$$

$\sqrt{\frac{1}{4}} = \frac{1}{2}$

$$= \frac{\sqrt{3}}{\sqrt{6} \cdot \sqrt{2}} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

best rank 1 approx. of A