SVD: $\underset{m\times n}{A} = U \underset{m\times n}{\Sigma} V^T$

- U, V orthogonal
- Σ diagonal entries $\sigma_i \geq 0$

Recipe:

- $A^TA = P \overset{\lambda_i}{D} P^T$ $\qquad A^TA = V\Sigma^T\Sigma V^T$

  $\Rightarrow V = P \qquad \sigma_i = \sqrt{\lambda_i}$ $\qquad U\Sigma = AV$

- $u_i = \frac{1}{\sigma_i} A v_i$

EG SVD of $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = A$

- $A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\qquad \lambda = \overset{\lambda_1}{3}, \overset{\lambda_2}{1}$

  3-eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ v_1

  1-eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ v_2

$\Rightarrow V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

- $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

  $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

  choose u_3 so that U orthogonal

  u_3 orthogonal to u_1, u_2

  $\Leftrightarrow u_3$ in $\text{null}\begin{pmatrix} -2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ basis: $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

  $u_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad U = \begin{pmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$

$A = U\Sigma V^T$