Singular value decomposition

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

- Σ diagonal with entries $\sigma_i \geq 0$
  singular values of A
- U, V orthogonal

How to compute?

- $\underset{\text{symmetric}}{A^TA} = \underset{\text{cols: } v_i}{P}\ \underset{\text{entries: } \lambda_i}{D}\ P^T$

$\Longrightarrow V = P, \quad \sigma_i = \sqrt{\lambda_i}$

- cols of U ~ $U\Sigma = AV$
  $u_i = \frac{1}{\sigma_i} A v_i$
  i-th col: $u_i \sigma_i = A v_i$

fill in other col's of U so that U is orthogonal

Notes:

- if A is symmetric
  $A = PDP^T$
  $\leadsto U = P, \Sigma = D, V = P$
- if $A = U\Sigma V^T$
  $\leadsto \underset{\text{symmetric}}{A^TA} = (V\Sigma^T U^T) U \Sigma V^T$
  $= V \boxed{\Sigma^T\Sigma} V^T$
  $n\times n$ diagonal entries σ_i^2

EG $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

- $A^TA = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $\quad \lambda = \underset{\lambda_1}{8}, \underset{\lambda_2}{2}$
  8-eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ v_1
  2-eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ v_2

$\Longrightarrow V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{8} & \\ & \sqrt{2} \end{bmatrix}$

- $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{16}}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{4}}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Longrightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = U\Sigma V^T$