Powers of matrices

warmup $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

- eigenspaces : 2-eigenspace = span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$
  3-eigenspace = span$\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$
- $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ $\qquad A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$$

EG If A has λ-eigenvector v, what about A^2?

$Av = \lambda v$

$A^2 v = A\underbrace{Av}_{\lambda v} = \lambda \underbrace{Av}_{\lambda v}$

$= \lambda^2 v$

A^2 has λ^2-eigenvector v.

in general: A^n has λ^n-eigenvector v.

THM If $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$.

- $A^2 = PD\cancel{P^{-1}P}DP^{-1} = PD^2P^{-1}$
- also works for negative n : $A^{-1} = PD^{-1}P^{-1}$

EG $A = \begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$ Compute A^n.

① $\underset{\text{work!}}{A = PDP^{-1}}$ with $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 10 & \\ & 5 \end{bmatrix}$

② $A^n = PD^nP^{-1} = \underbrace{\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 10^n & 0 \\ 0 & 5^n \end{bmatrix}}_{\begin{bmatrix} 10^n & -5^n \\ 4\cdot 10^n & 5^n \end{bmatrix}} \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} 10^n + 4\cdot 5^n & 10^n - 5^n \\ 4\cdot 10^n - 4\cdot 5^n & 4\cdot 10^n + 5^n \end{bmatrix}$$

check

- $n=1$ $\begin{bmatrix} 6 & 1 \\ 4 & 9 \end{bmatrix}$
- $n=0$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$