Orthogonal matrices

note $A^TA = I \iff A$ has orthonormal cols

entry i,j of A^TA

$= (\text{col } i \text{ of } A)\cdot(\text{col } j \text{ of } A)$

$\overset{\text{if } A \text{ has orthonormal cols}}{=} \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

recall: entry i,j of AB

$= (\text{row } i \text{ of } A)(\text{col } j \text{ of } B)$

note If Q has orthonormal cols, the matrix for (orth.) projecting onto $\text{col}(Q)$ is QQ^T.

in general: $Q(\underbrace{Q^TQ}_{\text{here: } = I})^{-1}Q^T$ if cols of Q lin. independent

in particular matrix for projecting onto $\underset{n\times 1}{q}$ if $\|q\| = 1$ $= \underset{n\times n}{\underset{n\times 1}{q}\,\underset{1\times n}{q^T}}$

proj. of v onto q $= \dfrac{q\cdot v}{\underbrace{q\cdot q}_{=1}}\, q = q\cdot v\, q = q\, q^T v$

EG matrix for projecting onto $\frac{1}{\sqrt{11}}\begin{bmatrix}1\\3\\1\end{bmatrix}$

$= \frac{1}{\sqrt{11}}\begin{bmatrix}1\\3\\1\end{bmatrix}\frac{1}{\sqrt{11}}\begin{bmatrix}1 & 3 & 1\end{bmatrix} = \frac{1}{11}\begin{bmatrix}1 & 3 & 1\\ 3 & 9 & 3\\ 1 & 3 & 1\end{bmatrix}$ rank $= 1$

DEF A is an orthogonal matrix

$\iff A$ is $n\times n$ and has orthonormal cols

$\iff \underset{n\times n}{A^T}\,\underset{n\times n}{A} = \underset{n\times n}{I}$

THM A orthogonal

$\iff A^{-1} = A^T$

THM A orthogonal $\implies \det(A) = \pm 1$

$A^TA = I$

$\underbrace{\det(A^T)}_{=\det(A)}\det(A) = \underbrace{\det(I)}_{=1}$

recall: $\det(AB) = \det(A)\det(B)$

$\rightsquigarrow (\det(A))^2 = 1$

EG $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ both orthogonal

$\det = 1$, $\det = -1$