QR decomposition

THM $A = Q R$

A: $m \times n$, rank n "full column rank", cols of A lin. independent

Q: $m \times n$ R: $n \times n$

Q orthonormal cols

R upper triangular, invertible

recipe proof!

- Gram-Schmidt orthonormalization on cols of A to get cols of Q
  $w_1, \dots, w_n$ $q_1, \dots, q_n$

- during Gram-Schmidt: $\tilde{q}_k = w_k - \{\}q_1 - \dots - \{\}q_{k-1}$ $+0 \cdot q_{k+1}$

  $\tilde{q}_k$: q_k pre-normalized

  $w_k = \{\}q_1 + \dots + \{\}q_{k-1} + \{\}q_k$

  entries in col k of R

"lazy" alternative:

$$R = Q^T A$$

Q orthonormal cols $\Leftrightarrow Q^T Q = I$

$$Q^T A = \cancel{Q^T Q} R = R$$

EG

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \\ \begin{matrix} w_1 & w_2 \end{matrix} \end{array}$$

$$\tilde{q}_1 = w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

normalized: $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\tilde{q}_2 = w_2 - \frac{w_2 \cdot \tilde{q}_1}{\tilde{q}_1 \cdot \tilde{q}_1} \tilde{q}_1 = \frac{3}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$$

$q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\Rightarrow Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} \\ 0 & 4/\sqrt{6} \end{bmatrix}$$

$$w_2 = \frac{1}{\sqrt{3}} q_1 + \frac{4}{\sqrt{6}} q_2$$

$$A = Q R$$