Gram-Schmidt orthogonalization

Gram-Schmidt

input: basis $w_1, w_2, \ldots$ for W
output: orthogonal basis $q_1, q_2, \ldots$ for W

- $q_1 = w_1$
- $q_2 = w_2 - \begin{pmatrix}\text{proj. of } w_2\\ \text{onto } q_1\end{pmatrix}$ ⟵ "error" of the proj. of w_2 onto q_1 ⇝ orthogonal to q_1 !
- $q_3 = w_3 \underbrace{- \begin{pmatrix}\text{proj. of } w_3\\ \text{onto } q_1\end{pmatrix} - \begin{pmatrix}\text{proj. of } w_3\\ \text{onto } q_2\end{pmatrix}}_{= -\begin{pmatrix}\text{proj. of } w_3\\ \text{onto span}\{q_1, q_2\}\end{pmatrix}}$ orthogonal!
- $q_4 = \ldots$

comments

- Gram-Schmidt orthonormalization
  numerically by computer: normalize each q_i when introduced
  by hand: normalize all q_i at the end
- $\text{span}\{q_1, \ldots, q_k\} = \text{span}\{w_1, \ldots, w_k\}$ for $k = 1, 2, \ldots$
  characterizes q_i uniquely up to scaling

EG Orthogonal basis for span $\left\{ \underset{w_1}{\begin{bmatrix}0\\3\\0\\0\end{bmatrix}}, \underset{w_2}{\begin{bmatrix}2\\1\\0\\0\end{bmatrix}}, \underset{w_3}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \right\}$

not orthogonal

$$q_1 = w_1 = \begin{bmatrix}0\\3\\0\\0\end{bmatrix}$$

$$q_2 = w_2 - \underbrace{\frac{w_2 \cdot q_1}{q_1 \cdot q_1} q_1}_{\text{proj. } w_2 \text{ onto } q_1} = \begin{bmatrix}2\\1\\0\\0\end{bmatrix} - \frac{3}{9}\begin{bmatrix}0\\3\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$$

$$q_3 = w_3 - \frac{w_3 \cdot q_1}{q_1 \cdot q_1} q_1 - \frac{w_3 \cdot q_2}{q_2 \cdot q_2} q_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{3}{9}\begin{bmatrix}0\\3\\0\\0\end{bmatrix} - \frac{2}{4}\begin{bmatrix}2\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

orthogonal basis: q_1, q_2, q_3

if desired, orthonormal basis: $\underset{\frac{1}{3}q_1}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$