Orthogonal bases

$v_1, v_2, \dots$
- span V
- are linearly independ.

DEF basis $v_1, v_2, \dots, v_n$ of V

- is orthogonal if $v_i \cdot v_j = 0$ for $i \neq j$
- is orthonormal if, in addition, $\|v_i\| = 1$

note Let B be the matrix with cols $v_1, \dots, v_n$.

$v_1, \dots, v_n$ orthonormal basis (of their span) $\iff B^T B = I$

note Orthogonal vectors $\neq \vec{0}$ are linearly independent.

EG bases of $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

length $= \sqrt{2} = \sqrt{1^2 + (-1)^2 + 0^2}$

orthogonal basis

not orthonormal

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

length $= 1$

orthonormal basis

THM orthogonal proj. $\hat{w}$ of w onto V if $v_1, v_2, \dots, v_n$ is an orthogonal basis of V

$$= \underbrace{\frac{w \cdot v_1}{v_1 \cdot v_1} v_1}_{\text{orth. proj. of } w \text{ onto } v_1} + \dots + \underbrace{\frac{w \cdot v_n}{v_n \cdot v_n} v_n}_{\text{orth. proj. of } w \text{ onto } v_n}$$

Why? $\underbrace{\hat{w} - w}_{\text{error}}$ needs to be orthogonal to V

i.e. orthogonal to each v_i

$$(\hat{w} - w) \cdot v_i = \hat{w} \cdot v_i - w \cdot v_i$$

$$= \frac{w \cdot v_1}{v_1 \cdot v_1} \underbrace{v_1 \cdot v_i}_{=0 \text{ except if } i=1} + \dots + \frac{w \cdot v_n}{v_n \cdot v_n} \underbrace{v_n \cdot v_i}_{=0 \text{ except if } i=n} - w \cdot v_i$$

$$= \frac{w \cdot v_i}{\cancel{v_i \cdot v_i}} \cancel{v_i \cdot v_i} - w \cdot v_i = 0$$

important $v_1, v_2, \dots, v_n$ orthogonal basis of V. w in V.

$$\Longrightarrow w = \underbrace{\frac{w \cdot v_1}{v_1 \cdot v_1} v_1}_{\text{orth. proj. of } w \text{ onto } v_1} + \dots + \underbrace{\frac{w \cdot v_n}{v_n \cdot v_n} v_n}_{\text{orth. proj. of } w \text{ onto } v_n}$$