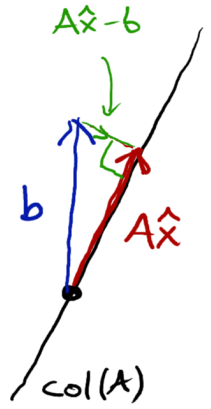
Least squares

EG $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (A, b)

has no solution x
because b is not in $\text{col}(A)$



DEF $\hat{x}$ least squares solution of $Ax = b$

$\overset{\text{DEF}}{\Longleftarrow}$ $A\hat{x} - b$ is as small as possible
(error; minimal norm)

$\Longleftrightarrow$ $A\hat{x} - b$ is orthogonal to $\text{col}(A)$

$\Longleftrightarrow$ $A\hat{x} - b$ is in $\text{null}(A^T)$

$\Longleftrightarrow$ $A^T(A\hat{x} - b) = 0$

$\Longleftrightarrow$ $A^TA\hat{x} = A^Tb$ **normal equations**

FTLA
$\text{col}(A)^\perp = \text{null}(A^T)$

EG $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (A, b)

$$A^TA = \underset{2\times3}{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 10 & 5 \\ 5 & 30 \end{bmatrix}} \qquad A^Tb = \underset{2\times1}{\begin{bmatrix} 4 \\ 8 \end{bmatrix}} \quad (2\times3,\ 3\times1)$$

normal equations:

$$\begin{bmatrix} 10 & 5 \\ 5 & 30 \end{bmatrix} \hat{x} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$(A^TA)^{-1}$

$$\hat{x} = \frac{1}{275} \begin{bmatrix} 30 & -5 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{55} \begin{bmatrix} 16 \\ 12 \end{bmatrix}$$

least squares solution

check $A\hat{x} - b = \frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ needs to be orthogonal to $\text{col}(A)$

i.e. $\frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 0$ ✓ $\quad \frac{1}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = 0$ ✓