Fundamental theorem of linear algebra

fundamental subspaces $\text{col}(A)$, $\text{row}(A)$, $\text{null}(A)$, $\text{null}(A^T)$ "left null space of A"

$= \text{col}(A^T)$

x in $\text{null}(A)$ $\iff Ax = 0$

x in $\text{null}(A^T)$ $\iff A^Tx = 0$ $\iff x^TA = 0^T$

$(AB)^T = B^TA^T$ $(A^T)^T = A$

THM FTLA — A $m \times n$, rank r (number of pivots in RREF of A)

	dim	subspace of	orthogonal complement of
$\text{col}(A)$	r	$\mathbb{R}^m$	$\text{null}(A^T)$
$\text{row}(A)$	r	$\mathbb{R}^n$	$\text{null}(A)$
$\text{null}(A)$	$n-r$	$\mathbb{R}^n$	$\text{row}(A)$
$\text{null}(A^T)$	$m-r$	$\mathbb{R}^m$	$\text{col}(A)$

EG $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 3 \end{bmatrix}$ RREF $\rightsquigarrow \begin{bmatrix} \textcircled{1} & 2 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A) = 2$

	dim	basis
$\text{col}(A)$	2	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$\text{row}(A)$	2	$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$
$\text{null}(A)$	$4-2=2$	$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ $\leftarrow \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix}$ relations between cols of A
$\text{null}(A^T)$	$3-2=1$	$\begin{bmatrix} 0 \\ -3/2 \\ 1 \end{bmatrix}$ relation between rows of A

extra work