Orthogonal complements

DEF V subspace of $\mathbb{R}^n$

$V^\perp$ = **orthogonal complement** of V

= all vectors (from $\mathbb{R}^n$) that are orthogonal to every vector in V

$v \perp w$
$\iff$ v, w orthogonal
$\iff$ $v \cdot w = 0$

EG $V = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

$V^\perp = \operatorname{span}\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$

$\operatorname{null}\left(\begin{bmatrix} 2 & 3 \end{bmatrix}\right)$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \perp \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\iff 2x_1 + 3x_2 = 0$

$\iff \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

EG $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$

$V^\perp = \operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \right)$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \perp \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$\iff x_1 + 2x_2 + x_3 = 0$
and $3x_1 + x_2 + 2x_3 = 0$

$\iff \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\xrightarrow[\text{work}]{}$ basis for $V^\perp$:

$$\begin{bmatrix} -3/5 \\ -1/5 \\ 1 \end{bmatrix}$$

check: $\begin{bmatrix} -3/5 \\ -1/5 \\ 1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ✓

$\perp \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ ✓

$V = \operatorname{col}(A) \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\implies V^\perp = \operatorname{null}(A^T)$

THM $\operatorname{col}(A)^\perp = \operatorname{null}(A^T)$