Crash course: eigenvalues + eigenvectors

A $n \times n$

$$Ax = \lambda x \quad \text{with } x \neq 0$$

$\Longrightarrow$ x is eigenvector with eigenvalue λ

key observation

$$Ax - \lambda x = 0$$
$$(A - \lambda I)x = 0$$

typically only has solution $x = 0$

unless $\det(A - \lambda I) = 0$

recipe to find eigenvalues + eigenvectors of A

① solve $\underbrace{\det(A - \lambda I)}_{\text{degree } n \text{ polynomial}} = 0$: char poly of A

$\Longrightarrow$ eigenvalues λ

② for each λ, find corresponding eigenvectors x by solving $(A - \lambda I)x = 0$

EG $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

① char. poly: $|A - \lambda I| = \begin{vmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{vmatrix}$

$$= (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6$$
$$= (\lambda - 3)(\lambda + 2)$$

eigenvalues: $\lambda = 3, -2$

② $\lambda = 3$: $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} x = \vec{0}$ $\qquad 5x_1 - 10x_2 = 0$

$\Longrightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ or any multiple

$\lambda = -2$: $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} x = \vec{0}$ $\qquad \begin{aligned} 10x_1 - 10x_2 &= 0 \\ 5x_1 - 5x_2 &= 0 \end{aligned}$

$\Longrightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple

check $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is 3-eigenvector

$$\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 8 + 1 \cdot (-10) \\ 2 \cdot 5 + 1 \cdot (-7) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector

$$\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$