

# Midterm #1

Please print your name:

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No notes, calculators or tools of any kind are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

## Problem 1. (6 points)

(a) Find the least squares solution to  $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ -1 \end{bmatrix}$ .

(b) Determine the least squares line for the data points  $(-1, 2), (0, 4), (1, 2), (1, -1)$ .

**Solution.** Let  $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ -1 \end{bmatrix}$ .

(a) Since  $A^T A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$  and  $A^T \mathbf{b} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$ , so the normal equations  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$  are

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

Solving, we find that the least squares solution is  $\hat{\mathbf{x}} = \frac{1}{11} \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

(b) We need to determine the values  $a, b$  for the least squares line  $y = a + bx$ . The equations  $a + bx_i = y_i$  translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ -1 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

Hence, the least squares line is  $y = 2 - x$ .

**Problem 2. (9 points)**

- (a) Using Gram–Schmidt, obtain an orthonormal basis for  $W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$ .
- (b) Determine the orthogonal projection of  $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$  onto  $W$ .
- (c) Determine the  $QR$  decomposition of the matrix  $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ .
- (d) Determine a basis for the orthogonal complement  $W^\perp$ .

**Solution.**

- (a) Let  $w_1, w_2$  be the vectors spanning  $W$ . We first construct an orthogonal basis  $q_1, q_2$  using Gram–Schmidt (and then normalize afterwards):

- $q_1 = w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- $q_2 = w_2 - \frac{w_2 \cdot q_1}{q_1 \cdot q_1} q_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Normalizing, we obtain the orthonormal basis  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  for  $W$ .

- (b) The orthogonal projection of  $v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$  onto  $W$  is

$$\frac{v \cdot q_1}{q_1 \cdot q_1} q_1 + \frac{v \cdot q_2}{q_2 \cdot q_2} q_2 = \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

(Check: the error  $\frac{2}{3}(1, -2, -1)^T$  is indeed orthogonal to  $W$ .)

- (c) From the first part, we know that  $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$ .

$$\text{Hence, } R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

- (d) Clearly,  $\dim W^\perp = 1$ , so that  $W^\perp$  is spanned by a single vector.

One way to determine vectors  $W^\perp$  is to take any vector  $v$  (not in  $W$ ) and project  $v$  onto  $W$ . The error of that projection then is in  $W^\perp$ .

Without extra computation, we can therefore take the error of the projection in the second part of this problem.

$$\text{Indeed, the vector } \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \text{ is a basis for } W^\perp.$$

**Problem 3. (3 points)** We want to find values for the parameters  $a, b, c$  such that  $z = a\sqrt{x} + b + cy^2$  best fits some given points  $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots$ . Set up a linear system such that  $[a, b, c]^T$  is a least squares solution.

**Solution.** The equations  $a\sqrt{x_i} + b + cy_i^2 = z_i$  translate into the system:

$$\underbrace{\begin{bmatrix} \sqrt{x_1} & 1 & y_1^2 \\ \sqrt{x_2} & 1 & y_2^2 \\ \sqrt{x_3} & 1 & y_3^2 \\ \vdots & \vdots & \vdots \end{bmatrix}}_A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_z$$

Of course, this is usually inconsistent. To find the best possible  $a, b, c$  we compute a least squares solution.

**Problem 4. (3 points)** Let  $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ .

(a) A basis for  $\text{null}(A)$  is . A basis for  $\text{col}(A)$  is .

(b)  $\dim \text{col}(A) =$  ,  $\dim \text{row}(A) =$  ,  $\dim \text{null}(A) =$  ,  $\dim \text{null}(A^T) =$  .

**Solution.**

(a) A basis for  $\text{null}(A)$  is  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$ . A basis for  $\text{col}(A)$  is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

(b)  $\dim \text{col}(A) = 2, \dim \text{row}(A) = 2, \dim \text{null}(A) = 2, \dim \text{null}(A^T) = 0$

**Problem 5. (9 points)** Fill in the blanks.

(a)  $\text{null}(A)$  is the orthogonal complement of .  $\text{row}(A)$  is the orthogonal complement of .

(b) If  $A$  is a  $5 \times 7$  matrix with rank 4, then  $\dim \text{col}(A) =$   and  $\dim \text{null}(A) =$  .

(c)  $\hat{x}$  is a least squares solution of  $Ax = b$  if and only if .

(d) By definition, a matrix  $Q$  is orthogonal if and only if .

- (e) If  $Q$  is orthogonal, then  $\det(Q)$  is .
- (f) The linear system  $A\mathbf{x} = \mathbf{b}$  is consistent if and only if  $\mathbf{b}$  is orthogonal to .
- (g) If  $A$  is a symmetric  $2 \times 2$  matrix with 3-eigenvector  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$  and  $\det(A) = 6$ .  
Then  $A$  has -eigenvector .
- (h) The projection matrix for orthogonally projecting onto  $\text{col}(A)$  is . [A has independent columns.]  
If  $A$  is orthogonal, this simplifies to .
- (i) If  $W$  is the subspace of  $\mathbb{R}^6$  of all solutions to  $x_1 + 3x_4 - 2x_6 = 0$ , then  $\dim W =$  ,  $\dim W^\perp =$  .

**Solution.**

- (a)  $\text{null}(A)$  is the orthogonal complement of  $\text{row}(A)$ .  $\text{row}(A)$  is the orthogonal complement of  $\text{null}(A)$ .
- (b) If  $A$  is a  $5 \times 7$  matrix with rank 4, then  $\dim \text{col}(A) = 4$  and  $\dim \text{null}(A) = 7 - 4 = 3$ .
- (c)  $\hat{\mathbf{x}}$  is a least squares solution of  $A\mathbf{x} = \mathbf{b}$  if and only if  $A^T A\mathbf{x} = A^T \mathbf{b}$ .
- (d) By definition, a matrix  $Q$  is orthogonal if and only if  $Q$  is  $n \times n$  (square) and  $Q$  has orthonormal columns.
- (e) If  $Q$  is orthogonal, then  $\det(Q)$  is  $\pm 1$ .
- (f) The linear system  $A\mathbf{x} = \mathbf{b}$  is consistent if and only if  $\mathbf{b}$  is orthogonal to  $\text{null}(A^T)$ .
- (g) If  $A$  is a symmetric  $2 \times 2$  matrix with 3-eigenvector  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$  and  $\det(A) = 6$ .  
Then  $A$  has 2-eigenvector  $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ .
- (h) The projection matrix for orthogonally projecting onto  $\text{col}(A)$  is  $P = A(A^T A)^{-1} A^T$ .  
If  $A$  has orthonormal columns (so that  $A^T A = I$ ), this simplifies to  $AA^T$ .  
If  $A$  is orthogonal, this further simplifies to  $I$ .
- (i)  $\dim W = 6 - 1 = 5$  and  $\dim W^\perp = 6 - 5 = 1$ .

(extra scratch paper)