# Preparing for Midterm #1

**MATH 316 — Linear Algebra II**
**Friday, Feb 21**

***Please print your name:***

**Problem 1.**

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto $W$.

(c) Determine the $QR$ decomposition of the matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(d) Determine a basis for the orthogonal complement $W^{\perp}$.

**Problem 2.**

(a) Find the least squares solution to the system $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto the space $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$?

(c) Determine the least squares line for the data points $(-2, 1), (-1, 0), (0, 3), (2, 1)$.

(d) Determine the projection matrix $P$ for orthogonally projecting onto $W$.

**Problem 3.** A scientist tries to find the relation between the mysterious quantities $x$ and $y$.

She measures the following values:

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 |
| $y$ | 2 | 5 | 9 | 17 |

(a) Our scientist has reason to expect that $y$ is a linear function of the form $a + bx$. Find the best estimate for the coefficients. ["best" in the least squares sense]

(b) What changes if we suppose that $y$ is a quadratic function of the form $a + bx + cx^2$? Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

**Problem 4.**

(a) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$ as $A = PDP^T$. (That is, find the matrices $P$ and $D$.)

(b) Let $A$ be a symmetric $2 \times 2$ matrix with 2-eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\det(A) = -6$. Diagonalize $A$.

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**Problem 5.**

(a) Is it true that $A^T A$ is always symmetric?

(b) If the columns of $A$ are orthogonal, what can you say about $A^T A$?

(c) Note that $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Why is it incorrect that the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto span$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ is $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$? Explain!

(d) For which matrices $A$ is it true that $A^{-1} = A^T$?

**Problem 6.**

(a) We want to find values for the parameters $a, b, c$ such that $y = a + bx + \frac{c}{x}$ best fits some given points $(x_1, y_1)$, $(x_2, y_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

(b) We want to find values for the parameters $a, b$ such that $y = (a + bx)e^x$ best fits some given points $(x_1, y_1)$, $(x_2, y_2), \ldots$ Set up a linear system such that $[a, b]^T$ is a least squares solution.

(c) We want to find values for the parameters $a, b, c$ such that $z = a + bx - c\sqrt{y}$ best fits some given points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

**Problem 7.** Let $W$ be the subspace of $\mathbb{R}^4$ of all solutions to $x_1 + x_2 + x_3 - x_4 = 0$.

(a) Find a basis for $W$.

(b) Find a basis for the orthogonal complement $W^\perp$.

(c) Determine the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto $W^\perp$.

(d) Determine the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto $W$.

**Problem 8.** Suppose that $A$ is a $3 \times 5$ matrix of rank 3.

(a) For each of the four fundamental subspaces of $A$, state which space it is a subspace of.

(b) What are the dimensions of all four fundamental subspaces?

(c) Which fundamental subspaces are orthogonal complements of each other?

(d) For the specific matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$, compute a basis for each fundamental subspace.

(e) Observe that $\operatorname{rank}(A) = 3$. Then, verify that all your predictions made in the first three parts do in fact hold.

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