Problem 3

Example 22. Find the best approximation of $f(x) = x^2$ on the interval $[1, 4]$ using a function of the form $y = a\sqrt{x}$.

Solution. Because we are working with functions on $[1, 4]$, the dot product between two functions is

$$\langle f(x), g(x) \rangle = \int_1^4 f(t) g(t) \mathrm{d}t.$$

The best approximation is the orthogonal projection of x^2 onto $\operatorname{span}\{\sqrt{x}\}$, which is

$$\begin{aligned}\frac{\langle x^2, \sqrt{x} \rangle}{\langle \sqrt{x}, \sqrt{x} \rangle} \sqrt{x} &= \frac{\int_1^4 t^2 \cdot \sqrt{t} \mathrm{d}t}{\int_1^4 \sqrt{t} \cdot \sqrt{t} \mathrm{d}t} \cdot \sqrt{x} = \frac{\int_1^4 t^{5/2} \mathrm{d}t}{\int_1^4 t \mathrm{d}t} \cdot \sqrt{x} \\ &= \frac{\left[\frac{1}{7/2} t^{7/2}\right]_1^4}{\left[\frac{1}{2} t^2\right]_1^4} \cdot \sqrt{x} = \frac{\frac{2}{7} \cdot 4^{7/2} - \frac{2}{7}}{\frac{1}{2} \cdot 4^2 - \frac{1}{2}} \cdot \sqrt{x} = \frac{\frac{254}{7}}{\frac{15}{2}} \cdot \sqrt{x} = \frac{508}{105} \sqrt{x}.\end{aligned}$$