Problem 2

Example 15. Compute the SVD of $A = \begin{bmatrix} -6 & 2 \\ 6 & -2 \end{bmatrix}$.

Solution. (by hand; you will need to show all steps on the final exam)

- First, we need to diagonalize $A^T A = \begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -6 & 2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 72 & -24 \\ -24 & 8 \end{bmatrix}$.
  $\det\left(\begin{bmatrix} 72-\lambda & -24 \\ -24 & 8-\lambda \end{bmatrix}\right) = (72-\lambda)(8-\lambda) - 576 = \lambda^2 - 80\lambda = \lambda(\lambda - 80)$
  Hence, the eigenvalues of $A^T A$ are $0, 80$.
  - $\lambda = 0$: $\begin{bmatrix} 72 & -24 \\ -24 & 8 \end{bmatrix} \overset{R_2 + \frac{1}{3}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 72 & -24 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{72}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$
    Hence, the 0-eigenspace has basis $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
  - $\lambda = 80$: $\begin{bmatrix} -8 & -24 \\ -24 & -72 \end{bmatrix} \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -8 & -24 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{8}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$
    Hence, the 80-eigenspace has basis $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

  Thus $A^T A = PDP^T$ with $D = \begin{bmatrix} 80 & \\ & 0 \end{bmatrix}$ and $P = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.
  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^T A = V\Sigma^2 V^T$, we conclude that $V = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{80} & \\ & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{80}}\begin{bmatrix} -6 & 2 \\ 6 & -2 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{800}}\begin{bmatrix} 20 \\ -20 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
  We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.
  For instance, $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so that $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{80} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution. (using Sage) We obtain the same solution (up to a sign in U and V):

```
>>> A = matrix(RDF, [[-6,2],[6,-2]])
>>> U,S,V = A.SVD()
>>> U
```

$$\begin{pmatrix} -0.7071067811865472 & 0.7071067811865472 \\ 0.7071067811865472 & 0.7071067811865475 \end{pmatrix}$$

```
>>> S
```

$$\begin{pmatrix} 8.944271909999161 & 0.0 \\ 0.0 & 2.1065000811460205 \times 10^{-16} \end{pmatrix}$$

```
>>> V
```

$$\begin{pmatrix} 0.9486832980505138 & -0.31622776601683783 \\ -0.31622776601683783 & -0.9486832980505138 \end{pmatrix}$$

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Problem 3

Example 16. Compute the SVD of $A = \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix}$.

Solution. (by hand; you will need to show all steps on the final exam)

- First, we need to diagonalize $A^TA = \begin{bmatrix} -7 & 5 & 1 \\ -1 & -5 & 3 \end{bmatrix} \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 75 & -15 \\ -15 & 35 \end{bmatrix}$.

  $\det\left(\begin{bmatrix} 75-\lambda & -15 \\ -15 & 35-\lambda \end{bmatrix}\right) = (75-\lambda)(35-\lambda) - 225 = \lambda^2 - 110\lambda + 2400 = (\lambda - 30)(\lambda - 80)$

  Hence, the eigenvalues of A^TA are $30, 80$.

  - $\lambda = 30$: $\begin{bmatrix} 45 & -15 \\ -15 & 5 \end{bmatrix} \overset{R_2 + \frac{1}{3}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 45 & -15 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{45}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$

    Hence, the 30-eigenspace has basis $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

  - $\lambda = 80$: $\begin{bmatrix} -5 & -15 \\ -15 & -45 \end{bmatrix} \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -5 & -15 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{5}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

    Hence, the 80-eigenspace has basis $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

  Thus $A^TA = PDP^T$ with $D = \begin{bmatrix} 80 & \\ & 30 \end{bmatrix}$ and $P = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA = V\Sigma^2 V^T$, we conclude that $V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{30} \\ 0 & 0 \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{80}} \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{800}} \begin{bmatrix} 20 \\ -20 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

  Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -7 & -1 \\ 5 & -5 \\ 1 & 3 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{300}} \begin{bmatrix} -10 \\ -10 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

  We cannot obtain $\boldsymbol{u}_3$ like this because there is no σ_3. We need to choose $\boldsymbol{u}_3$ so that U is orthogonal.

  To find a vector that is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we compute:

  $$\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \overset{R_2 + R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \overset{-\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \overset{R_1 + R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

  Therefore, $\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

  Normalizing $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ to $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, we conclude that $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{30} \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution. (using Sage) We obtain the same solution (up to a sign in U and V):

```
>>> A = matrix(RDF, [[-7,-1],[5,-5],[1,3]])
```

```
>>> U,S,V = A.SVD()
```

```
>>> U
```

$$\begin{pmatrix} -0.7071067811865476 & -0.577350269189626 & 0.40824829046386296 \\ 0.7071067811865477 & -0.5773502691896258 & 0.4082482904638629 \\ -1.4525337733367862 \times 10^{-17} & 0.5773502691896257 & 0.816496580927726 \end{pmatrix}$$

```
>>> S
```

$$\begin{pmatrix} 8.94427190999157 & 0.0 \\ 0.0 & 5.477225575051662 \\ 0.0 & 0.0 \end{pmatrix}$$

```
>>> V
```

$$\begin{pmatrix} 0.9486832980505138 & 0.316227766016838 \\ -0.316227766016838 & 0.9486832980505138 \end{pmatrix}$$

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