Homework Set 9 (Lecture 28)

Problem 1

Example 14. Compute the SVD of $A = \begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix}$. That is, decompose A as $A = U\Sigma V^T$.

Solution. (by hand; you will need to show all steps on the final exam)

- First, we need to diagonalize $A^TA = \begin{bmatrix} -3 & -5 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 34 & -12 \\ -12 & 16 \end{bmatrix}$.
  $\det\left(\begin{bmatrix} 34-\lambda & -12 \\ -12 & 16-\lambda \end{bmatrix}\right) = (34-\lambda)(16-\lambda) - 144 = \lambda^2 - 50\lambda + 400 = (\lambda-10)(\lambda-40)$
  Hence, the eigenvalues of A^TA are $10, 40$.
  - $\lambda = 10$: $\begin{bmatrix} 24 & -12 \\ -12 & 6 \end{bmatrix} \overset{R_2+\frac{1}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 24 & -12 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{24}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$
    Hence, the 10-eigenspace has basis $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ or, easier for working by hand, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
  - $\lambda = 40$: $\begin{bmatrix} -6 & -12 \\ -12 & -24 \end{bmatrix} \overset{R_2-2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} -6 & -12 \\ 0 & 0 \end{bmatrix} \overset{-\frac{1}{6}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$
    Hence, the 40-eigenspace has basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  Thus $A^TA = PDP^T$ with $D = \begin{bmatrix} 40 & \\ & 10 \end{bmatrix}$ and $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$.
  [We have to normalize the eigenvectors! Otherwise, we would only have a diagonalization PDP^{-1}.]

- Since $A^TA = V\Sigma^2V^T$, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{40} & \\ & \sqrt{10} \end{bmatrix}$.

- From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{40}}\begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{200}}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 4 \\ -5 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{50}}\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{40} & \\ & \sqrt{10} \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution. (using Sage) We obtain the same solution (up to a sign in U and V):

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>>> A = matrix(RDF, [[-3,4],[-5,0]])
>>> U,S,V = A.SVD()
>>> U
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$$\begin{pmatrix} -0.7071067811865475 & -0.7071067811865476 \\ -0.7071067811865476 & 0.7071067811865475 \end{pmatrix}$$

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>>> S
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$$\begin{pmatrix} 6.324555320336758 & 0.0 \\ 0.0 & 3.16227766016838 \end{pmatrix}$$

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>>> V
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$$\begin{pmatrix} 0.8944271909999159 & -0.44721359549995804 \\ -0.44721359549995804 & -0.8944271909999159 \end{pmatrix}$$

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