Review.

- Let A be $n \times n$. The matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Then, $\frac{\mathrm{d}}{\mathrm{d}t} e^{At} = A e^{At}$.

Why? $\frac{\mathrm{d}}{\mathrm{d}t} e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$

- If $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

Example 134. The matrix exponential shares many other properties of the usual exponential:

- $e^Ae^B = e^{A+B} = e^Be^A$ if $AB = BA$

Why the condition $\boldsymbol{AB = BA}$? By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$ In order to simplify that to

$$e^Ae^B = \left(I + A + \frac{A^2}{2!} + \ldots\right)\left(I + B + \frac{B^2}{2!} + \ldots\right),$$

we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- e^A is invertible and $(e^A)^{-1} = e^{-A}$

Why? That actually follows from the previous property.

Example 135. Compute e^{At} for $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$.

Solution.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$. Note that $2I$ and N commute.

  Hence, $e^{At} = e^{2It + Nt} = e^{2It}e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!}N^2 + \ldots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It}e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ & e^{2t} \end{bmatrix}$.

Advanced. Can you show that $A^n = \begin{bmatrix} 2^n & n2^{n-1} \\ & 2^n \end{bmatrix}$?

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Example 136. Solve the differential equation

$$\boldsymbol{y}' = \underbrace{\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}}_{A} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_0}$$

Solution. Repeating the work in the previous example, the solution to the differential equation is

$$\begin{aligned} \boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 \\ &= e^{2It+Nt}\boldsymbol{y}_0 \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= e^{2It}e^{Nt}\boldsymbol{y}_0 \qquad (\text{because } 2It \text{ and } Nt \text{ commute}) \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\left(1 + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots\right)\boldsymbol{y}_0 \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}(1 + Nt)\boldsymbol{y}_0 \qquad (\text{because } N^2 = \boldsymbol{0}) \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix}. \end{aligned}$$

Check. We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$.
Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

Comment. For applications, having solutions like $te^{\lambda t}$ or $t\cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.

Important comment. Note that we can immediately see from the solution that the original matrix A is not diagonalizable: there is a term te^{2t}, whereas in the diagonalizable case we would only see exponentials like e^{2t} by themselves.

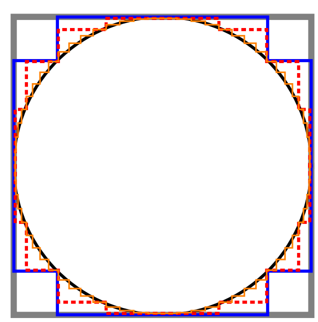
In our upcoming discussion of complex numbers we will see that e^{2it} (here, $2i$ would be the eigenvalue) can be rewritten in terms of $\cos(2t)$ and $\sin(2t)$. Both of these are periodic and bounded, so that the same is true for any linear combination.

In that case, if the eigenvalue $2i$ was repeated in such a way that the matrix A is not diagonalizable, then we would get the functions $t\cos(2t)$ and $t\sin(2t)$ in our solutions. These, however, are not bounded! This phenomenon (getting solutions that are unbounded under the right/wrong circumstances) is called **resonance**.

https://en.wikipedia.org/wiki/Resonance

Understanding when resonance occurs is of crucial importance for practical applications.

Remark 137. (April Fools' Day!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

Comment. We'll actually come back to this. It's related to linear algebra in infinite dimensions.