

**Definition 116.** Let  $A$  be  $n \times n$ . The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

**Why?** As a consequence of this definition (which is the motivation for that definition in the first place),

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt}\left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots\right] \\ &= 0 + A + A^2t + \frac{1}{2!}A^3t^2 + \dots = Ae^{At}. \end{aligned}$$

Therefore,  $\mathbf{y}(t) = e^{At}\mathbf{y}_0$  indeed solves the initial value problem  $\mathbf{y}' = A\mathbf{y}$ ,  $\mathbf{y}(0) = \mathbf{y}_0$ .

**How to actually compute  $e^A$ ?** Well, this Taylor series involves the powers  $A^n$  of  $A$ . How would you compute, say,  $A^{100}$ ? The answer is diagonalization!

**Theorem 117.** Suppose  $A = PDP^{-1}$ . Then,  $e^A = Pe^DP^{-1}$ .

**Why?** Recall that, if  $A = PDP^{-1}$ , then  $A^n = PD^nP^{-1}$ .

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \dots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \dots\right)P^{-1} = Pe^DP^{-1} \end{aligned}$$

**Comment.** By the same argument, if  $A = PDP^{-1}$ , then  $f(A) = Pf(D)P^{-1}$  for any “nice” function  $f$ . Here, “nice” means that  $f$  has a convergent Taylor series  $f(x) = \sum_{n \geq 0} a_n x^n$ .

More explicitly, if  $A = P \operatorname{diag}(\lambda_1, \dots, \lambda_n) P^{-1}$ , then  $f(A) = P \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n)) P^{-1}$ .

**Example 118.** If  $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ , then  $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$ .

**Example 119.** If  $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ , then  $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \dots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$ .

Clearly, this works to obtain  $e^D$  for any diagonal matrix  $D$ .

In particular, for  $At = \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix}$ ,  $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \dots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$ .

**Example 120. (homework)** Diagonalize  $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ .

**Solution. (final solution only)**  $A = PDP^{-1}$  with  $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  and  $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$ .

**Example 121.** Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

**Solution.** Recall that the solution to  $\mathbf{y}' = A\mathbf{y}$ ,  $\mathbf{y}(0) = \mathbf{y}_0$  is  $\mathbf{y} = e^{At}\mathbf{y}_0$ .

- First, we diagonalize:

For  $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ ,  $A = PDP^{-1}$  with  $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  and  $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$ . (That's homework!)

- We can then compute the solution  $\mathbf{y}(t) = e^{At}\mathbf{y}_0$ :

$$\begin{aligned} \mathbf{y}(t) = e^{At}\mathbf{y}_0 &= Pe^{Dt}P^{-1}\mathbf{y}_0 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix} \end{aligned}$$

**Comment.** It is not necessary to compute  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$  (of course, you could do it, but that's more work).

Instead, recall that  $A^{-1}\mathbf{b}$  is the unique solution to  $A\mathbf{x} = \mathbf{b}$ . Here, solving  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ , we find  $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ .

**Check.**  $\mathbf{y} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$  indeed solves the original problem:

$$\mathbf{y}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} + 4e^{4t} \\ 4e^{4t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$