Homework Set 8 (Lecture 23)

Problem 1

Example 3. Compute $\exp(Dt)$ for the diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Solution. $e^{Dt} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$

Problem 2

Example 4. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 11 & 8 \\ -12 & -9 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Solution.

- First, we diagonalize $A = \begin{bmatrix} 11 & 8 \\ -12 & -9 \end{bmatrix}$:

  $\det\left(\begin{bmatrix} 11-\lambda & 8 \\ -12 & -9-\lambda \end{bmatrix}\right) = (11-\lambda)(-9-\lambda) + 96 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$

  Hence, the eigenvalues of A are $-1, 3$.

  - $\lambda = -1$: $\begin{bmatrix} 12 & 8 \\ -12 & -8 \end{bmatrix} \overset{R_2 + R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 12 & 8 \\ 0 & 0 \end{bmatrix} \overset{\frac{1}{12}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$

    Hence, the -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 12 & 8 \\ -12 & -8 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$.

  - $\lambda = 3$: The 3-eigenspace $\operatorname{null}\left(\begin{bmatrix} 8 & 8 \\ -12 & -12 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  In conclusion, $A = PDP^{-1}$ with $P = \begin{bmatrix} -2/3 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & \\ & 3 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \underbrace{\begin{bmatrix} -\frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^{3t} \end{bmatrix}}_{\begin{bmatrix} -\frac{2}{3}e^{-t} & -e^{3t} \\ e^{-t} & e^{3t} \end{bmatrix}} \underbrace{\frac{1}{-\frac{2}{3}+1} \begin{bmatrix} 1 & 1 \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 9 \\ -9 \end{bmatrix}} = \begin{bmatrix} -6e^{-t} + 9e^{3t} \\ 9e^{-t} - 9e^{3t} \end{bmatrix} \end{aligned}$$

Problem 3

Example 5. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} -5 & 0 & -3 \\ -6 & -2 & -6 \\ 6 & 0 & 4 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

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Solution.

- First, we diagonalize $A = \begin{bmatrix} -5 & 0 & -3 \\ -6 & -2 & -6 \\ 6 & 0 & 4 \end{bmatrix}$:

$$\underbrace{\det\left(\begin{bmatrix} -5-\lambda & 0 & -3 \\ -6 & -2-\lambda & -6 \\ 6 & 0 & 4-\lambda \end{bmatrix}\right)}_{\text{expand by 2nd column}} = (-2-\lambda)\det\left(\begin{bmatrix} -5-\lambda & -3 \\ 6 & 4-\lambda \end{bmatrix}\right)$$

$$= (-2-\lambda)((-5-\lambda)(4-\lambda)+18) = (-2-\lambda)(\lambda^2+\lambda-2)$$

$$= (-2-\lambda)(\lambda-1)(\lambda+2)$$

Hence, the eigenvalues of A are $1, -2, -2$.

  - $\lambda = 1$: $\begin{bmatrix} -6 & 0 & -3 \\ -6 & -3 & -6 \\ 6 & 0 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 + R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -6 & 0 & -3 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \overset{\substack{-\frac{1}{6}R_1 \Rightarrow R_1 \\ -\frac{1}{3}R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

    Hence, the -1-eigenspace has basis $\begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}$.

  - $\lambda = -2$: $\begin{bmatrix} -3 & 0 & -3 \\ -6 & 0 & -6 \\ 6 & 0 & 6 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{-\frac{1}{3}R_1 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

    Since, $\begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, the -2-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

In conclusion, $A = PDP^{-1}$ with $P = \begin{bmatrix} -1/2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & -2 & \\ & & -2 \end{bmatrix}$.

- Finally, we compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$:

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = Pe^{Dt}P^{-1}\boldsymbol{y}_0 = \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{-2t} & \\ & & e^{-2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Note that $\boldsymbol{x} = \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is equivalent to $\begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. We solve this to find $\boldsymbol{x}$:

It follows from $\left[\begin{array}{ccc|c} -\frac{1}{2} & 0 & -1 & 3 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 + 2R_1 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|c} -\frac{1}{2} & 0 & -1 & 3 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & -1 & 7 \end{array}\right] \overset{\substack{-2R_1 \Rightarrow R_1 \\ -R_3 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -6 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -7 \end{array}\right]$

$\overset{\substack{R_1 - 2R_3 \Rightarrow R_1 \\ R_2 - 2R_3 \Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -7 \end{array}\right]$ that $\boldsymbol{x} = \begin{bmatrix} 8 \\ 8 \\ -7 \end{bmatrix}$. Therefore:

$$\boldsymbol{y}(t) = \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^{-2t} & \\ & & e^{-2t} \end{bmatrix} \begin{bmatrix} 8 \\ 8 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8e^t \\ 8e^{-2t} \\ -7e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^t + 7e^{-2t} \\ -8e^t + 8e^{-2t} \\ 8e^t - 7e^{-2t} \end{bmatrix}$$