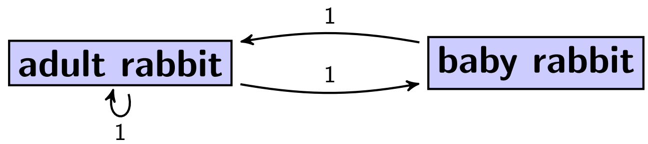
Review. Fibonacci numbers, Binet formula

Example 100. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let a_t be the number of adult rabbit pairs after t months. Likewise, b_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} a_t + b_t \\ a_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \end{bmatrix}.$$

That's precisely the transition for the Fibonacci numbers!

It follows that Fibonacci numbers count the number of rabbits in this model.

Comment. Note that the setup is very much as for Markov chains. Here, however, the outgoing values do not add to 100% for each state. Consequently, we cannot expect an equilibrium (and, indeed, the number of rabbits increases without bound).

Definition 101. A sequence a_n satisfying a recursion of the form

$$a_{n+d} = r_1 a_{n+d-1} + r_2 a_{n+d-2} + \ldots + r_d a_n$$

is called ***C*-finite** (or, **constant recursive**) of order d.

For instance. For the Fibonacci numbers, $d=2$ and $r_1 = r_2 = 1$.

In matrix-vector form.
$$\begin{bmatrix} a_{n+d} \\ a_{n+d-1} \\ \vdots \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 & r_2 & \cdots & r_{d-1} & r_d \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix}}_{T} \begin{bmatrix} a_{n+d-1} \\ a_{n+d-2} \\ \vdots \\ a_n \end{bmatrix}$$

By the same reasoning as for Fibonacci numbers, C-finite sequences have a Binet-like formula:

Theorem 102. (generalized Binet formula) Suppose the recursion matrix T has distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n + \ldots + C_d \lambda_d^n$$

for certain numbers $C_1, \ldots, C_d$.

For instance. For the Fibonacci numbers, $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$, and $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = -\frac{1}{\sqrt{5}}$.

Comment. A little more care is needed in the case that eigenvalues are repeated.

Armin Straub
straub@southalabama.edu

Corollary 103. Under the assumptions of the previous theorem, if λ_1 is the eigenvalue with the largest absolute value and $\lambda_1 > 0$, as well as $\alpha_1 \neq 0$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_1$.

Proof. This follows from $a_n = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n$ because, for large n, the term $C_1\lambda_1$ dominates the others. Indeed, we have

$$\frac{a_{n+1}}{a_n} = \frac{C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1} + \ldots + C_d\lambda_d^{n+1}}{C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_d\lambda_d^n} = \frac{C_1\lambda_1 + C_2\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\lambda_d\left(\frac{\lambda_d}{\lambda_1}\right)^n}{C_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \ldots + C_d\left(\frac{\lambda_d}{\lambda_1}\right)^n} \quad \stackrel{n\to\infty}{\longrightarrow} \quad \frac{C_1\lambda_1}{C_1} = \lambda_1.$$

□

Example 104. Consider the sequence a_n defined by $a_{n+3} = 4a_{n+2} - a_{n+1} - 6a_n$ and $a_0 = 0$, $a_1 = -2$, $a_2 = 2$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $0, -2, 2, 10, 50, 178, 602, 1930, 6050, \ldots$

Note that this sequence is C-finite of order 3.

(b) The recursion can be translated to $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

We expand by the first row: $\det\left(\begin{bmatrix} 4-\lambda & -1 & -6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = (4-\lambda)\lambda^2 - (-1)(-\lambda) - 6 = -\lambda^3 + 4\lambda^2 - \lambda - 6$

The eigenvalues of the transition matrix are the roots of this polynomial: $\lambda = -1, 2, 3$

[You will not be asked to find roots of cubic polynomials by hand.]

Hence, $a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot 3^n$ and we only need to figure out the two unknowns C_1, C_2, C_3.

Using the three initial conditions, we get three equations:

$(a_0 =)$ $C_1 + C_2 + C_3 = 0$, $(a_1 =)$ $-C_1 + 2C_2 + 3C_3 = -2$, $(a_2 =)$ $C_1 + 4C_2 + 9C_3 = 2$.

Solving, we find $C_1 = 1$, $C_2 = -2$ and $C_3 = 1$ so that, in conclusion, $a_n = (-1)^n - 2 \cdot 2^n + 3^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$.

Important comment. Right after computing the eigenvalues, we knew that this limit would be 3, except in the special (degenerate) case of $C_3 = 0$.

Example 105. (extra) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution. The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$ are $1 \pm \sqrt{5}$. Hence, $a_n = C_1(1+\sqrt{5})^n + C_2(1-\sqrt{5})^n$ for certain numbers C_1, C_2.

[Note that we cannot have $C_1 = 0$, because then $a_n = C_2(1-\sqrt{5})^n$ so that $a_0 = 0$ would imply $C_2 = 0$.]

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1}F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.