

Application: Fibonacci numbers

The numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0, F_1 = 1$.

How fast are they growing?

Have a look at ratios of Fibonacci numbers: $\frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.615, \frac{34}{21} = 1.619, \dots$

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\dots$

In other words, it appears that $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 98 below.

The crucial insight is the following simple observation:

$$F_{n+1} = F_n + F_{n-1} \quad \text{is equivalent to} \quad \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

In particular, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

Comment. Recurrence equations are discrete analogs of differential equations. We will later see the same idea applied when we reduce the order of a differential equation by introducing additional equations.

Everything we observe here about Fibonacci numbers holds for other sequences that satisfy similar recursion equations.

Theorem 98. (Binet's formula) $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$

Proof.

- We already observed that the recurrence $F_{n+1} = F_n + F_{n-1}$ translates into $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $T = PDP^{-1}$ with

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} &= T^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix} \end{aligned}$$

- Hence, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, which is the claimed formula.

□

Comment. For large n , $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n , it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \xrightarrow{n \rightarrow \infty} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

Comment. It follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Note that, given any Fibonacci-like recursion, we can apply our linear algebra skills in the same fashion. The next example illustrates how this is set up.

Example 99. Suppose the sequence a_n satisfies $a_{n+3} = 3a_{n+2} - 2a_{n+1} + 7a_n$. Write down a matrix-vector version of this recursion.

Solution.
$$\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$$