Example 87. (warmup) Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

- What are the eigenspaces?
- What are A^{-1} and A^{100}?

Solution.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 3-eigenvector. In other words, the 2-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and the 3-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.
- $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ and $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$

Comment. Algebraically, this looks like a very simple map. However, notice that it is not so easy to say what happens to, say, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ geometrically. That is because two things are happening: part of that vector is scaled by 2, the other part is scaled by 3.

Example 88. If A has λ-eigenvector $\boldsymbol{v}$, then what can we say about A^2?

Solution. A^2 has λ^2-eigenvector $\boldsymbol{v}$.
[Indeed, $A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v}$. This is even easier in words: multiplying $\boldsymbol{v}$ with A has the effect of scaling it by λ; hence, multiplying it with A^2 scales it by λ^2.]

Important comment. Similarly, A^{100} has λ^{100}-eigenvector $\boldsymbol{v}$.

Example 89. If a matrix A can be diagonalized as $A = PDP^{-1}$, what can we say about A^n?

Solution. First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$. Likewise, $A^n = PD^nP^{-1}$.

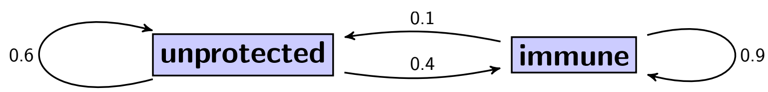
The point being that D^n is trivial to compute because D is diagonal.

In particular. $A^{-1} = PD^{-1}P^{-1}$

Application: Markov chains

Example 90. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.
What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)
y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $M = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.
It follows that M^2 describes the transition over 2 years. Likewise, M^n describes the transition over n years.
In particular, $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = M^n\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. Therefore, the powers of M are the key to understanding what happens in this model over time.

Armin Straub
straub@southalabama.edu