Sketch of Lecture 5

Fri, 1/24/2020

Example 17. (review) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Recall that $(AB)^T = B^T A^T$. This reflects the fact that, in the column-centric versus the row-centric interpretation of matrix multiplication, the order of the matrices is reversed.

Comment. When working with complex numbers, the fundamental role is not played by the transpose but by the **conjugate transpose** instead (we'll see that in our discussion of orthogonality): $A^* = \overline{A^T}$.

For instance, if $A = \begin{bmatrix} 1-3i & 5i \\ 2+i & 3 \end{bmatrix}$, then $A^* = \begin{bmatrix} 1+3i & 2-i \\ -5i & 3 \end{bmatrix}$.

Orthogonality

The inner product and distances

Definition 18. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 19. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

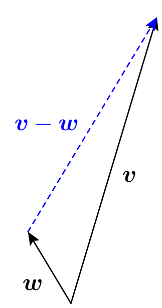
Definition 20.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 21. For instance, in $\mathbb{R}^2$, $\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 22. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

Comment. This is a vector version of $(x - y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

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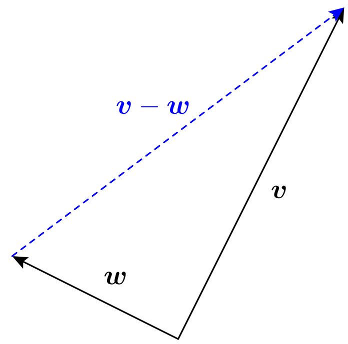
Definition 23. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v}\cdot\boldsymbol{w}=0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$\iff \|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2=\underbrace{\|\boldsymbol{v}-\boldsymbol{w}\|^2}_{\substack{=\|\boldsymbol{v}\|^2-2\boldsymbol{v}\cdot\boldsymbol{w}+\|\boldsymbol{w}\|^2 \\ \text{(by previous example)}}}$

$\iff -2\boldsymbol{v}\cdot\boldsymbol{w}=0$

$\iff \boldsymbol{v}\cdot\boldsymbol{w}=0$

Example 24. Determine a basis for the **orthogonal complement** of (the span of) $\begin{bmatrix}1\\2\\1\end{bmatrix}$.

What are we looking for? The orthogonal complement of $\boldsymbol{v}$ consists of all vectors that are orthogonal to $\boldsymbol{v}$. More generally, the orthogonal complement of a space V consists of all vectors that are orthogonal to every vector in V.

Solution. (staring/intution) We are working in 3-dimensional space and already have 1 vector. The vectors orthogonal to it lie in a $3-1=2$-dimensional space (a plane).

Two of the vectors orthogonal to $\begin{bmatrix}1\\2\\1\end{bmatrix}$ are $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Knowing that the orthogonal complement has dimension 2, we conclude that $\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}$ is a basis.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ is $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

[Note how the dimensions add up to the dimension of the entire space: $1+2=3$.]

Solution. (professional) $\begin{bmatrix}1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ (dot product!) is the same as $\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ (matrix product!).

Hence, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ is the same as $\operatorname{null}(\begin{bmatrix}1&2&1\end{bmatrix})$.

Computing a basis for $\operatorname{null}(\begin{bmatrix}1&2&1\end{bmatrix})$ is easy since $\begin{bmatrix}1&2&1\end{bmatrix}$ is already in RREF.

Note that the general solution to $\begin{bmatrix}1&2&1\end{bmatrix}\boldsymbol{x}=0$ is $\begin{bmatrix}-2s-t\\s\\t\end{bmatrix}=s\begin{bmatrix}-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

A basis for $\operatorname{null}(\begin{bmatrix}1&2&1\end{bmatrix})$ therefore is $\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}$. (Check that these are indeed orthogonal to $\begin{bmatrix}1\\2\\1\end{bmatrix}$!)

Example 25. Determine a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$.

Solution. We are looking for vectors $\boldsymbol{x}$ such that $\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ and $\begin{bmatrix}3 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$.

The two equations can be combined into a single one: $\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}$.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$ is the same as $\operatorname{null}\left(\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\end{bmatrix}\right)$.

It remains to compute a basis for that null space:

$$\begin{bmatrix}1 & 2 & 1\\3 & 1 & 2\end{bmatrix}\overset{R_2-3R_1\Rightarrow R_2}{\rightsquigarrow}\begin{bmatrix}1 & 2 & 1\\0 & -5 & -1\end{bmatrix}\quad\overset{\text{back-substitution}}{\rightsquigarrow}\quad\begin{bmatrix}-3/5s\\-1/5s\\s\end{bmatrix}$$

Hence, a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$ is $\begin{bmatrix}-3/5\\-1/5\\1\end{bmatrix}$.

Check. $\begin{bmatrix}-3/5\\-1/5\\1\end{bmatrix}$ is indeed orthogonal to both $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\\2\end{bmatrix}$.

Just to make sure. Why was it clear that the orthogonal complement is 1-dimensional?