## Review: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of $A$ with **eigenvalue** $\lambda$ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, $A$ needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \Longleftrightarrow\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \Longleftrightarrow\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of $A$:

(a) First, find the eigenvalues $\lambda$ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in $\lambda$, called the **characteristic polynomial** of $A$.

(b) Then, for each eigenvalue $\lambda$, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

More precisely, we find a basis of eigenvectors for the $\lambda$-**eigenspace** $\mathrm{null}(A - \lambda I)$.

**Example 14.** $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ has one eigenvector that is "easy" to see. Do you see it?

**Solution.** Note that $A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a $2$-eigenvector.

**Just for contrast.** Note that $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

Suppose that $A$ is $n \times n$ and has independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Then $A$ can be **diagonalized** as $A = PDP^{-1}$, where

- the columns of $P$ are the eigenvectors, and
- the diagonal matrix $D$ has the eigenvalues on the diagonal.

Such a diagonalization is possible if and only if $A$ has enough (independent) eigenvectors.

**Comment.** If you don't quite recall why these choices result in the diagonalization $A = PDP^{-1}$, note that the diagonalization is equivalent to $AP = PD$.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix $P$.

$$A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- In summary: $AP = PD$

Armin Straub
straub@southalabama.edu

**Example 15.** Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues and bases for the eigenspaces of $A$.

(b) Diagonalize $A$. That is, determine matrices $P$ and $D$ such that $A = PDP^{-1}$.

**Solution.**

(a) By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda)-2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity $2$) and $\lambda = 5$.

**Comment.** At this point, we know that we will find one eigenvector for $\lambda = 5$ (more precisely, the $5$-eigenspace definitely has dimension $1$). On the other hand, the $2$-eigenspace might have dimension $2$ or $1$. In order for $A$ to be diagonalizable, the $2$-eigenspace must have dimension $2$. (Why?!)

- The $5$-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right)$. Proceeding as in Example 12, we obtain

$$\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}.$$

  In other words, the $5$-eigenspace has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

- The $2$-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$. Proceeding as in Example 13, we obtain

$$\operatorname{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right) \overset{\text{RREF}}{=} \operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

  In other words, the $2$-eigenspace has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

  **Comment.** So, indeed, the $2$-eigenspace has dimension $2$. In particular, $A$ is diagonalizable.

(b) A possible choice is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

**Comment.** However, many other choices are possible and correct. For instance, the order of the eigenvalues in $D$ doesn't matter (as long as the same order is used for $P$). Also, for $P$, the columns can be chosen to be any other set of eigenvectors.

**Example 16. (extra practice)** Diagonalize, if possible, the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Solution.** For instance, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & & \\ & 2 & \\ & & 2 \end{bmatrix}$. $B$ is not diagonalizable.

For instance, $C = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$.