

### Example 6.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 & 3 \\ -16 & 5 & 6 \\ -25 & 8 & 9 \end{bmatrix}$$

Multiplication (on the right) with that “almost identity matrix” is performing the column operation  $C_1 - 4C_2 \Rightarrow C_1$  (i.e.  $-4$  times the second column is added to the first column).

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

Multiplication (on the left) with the same matrix is performing the row operation  $R_2 - 4R_1 \Rightarrow R_2$ .

**Comment (again).** The row operations we are doing during Gaussian elimination can all be realized by multiplying (on the left) with “almost identity matrices”.

These matrices are called **elementary matrices** (they are obtained by performing a single elementary row operation on an identity matrix).

Elementary matrices are **invertible** because elementary row operations are reversible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Example 7.** Let us do Gaussian elimination on  $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$  until we have an echelon form:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \Rightarrow R_2} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

As last class, the row operation can be encoded by multiplication with an “almost identity matrix”  $E$ :

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_E \underbrace{\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}}_U$$

Since  $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$  (no calculation needed!), this means that

$$A = E^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}.$$

We factored  $A$  as the product of a lower and an upper triangular matrix!

$A = LU$  is known as the **LU decomposition** of  $A$ .  
 $L$  is lower triangular,  $U$  is upper triangular.

If  $A$  is  $m \times n$ , then  $L$  is an invertible lower triangular  $m \times m$  matrix, and  $U$  is a usual **echelon form** of  $A$ . Every matrix  $A$  has a LU decomposition (after possibly swapping some rows of  $A$  first).

- The matrix  $U$  is just the echelon form of  $A$  produced during Gaussian elimination.
- The matrix  $L$  can be constructed, entry-by-entry, by simply recording the row operations used during Gaussian elimination. (No extra work needed!)

**Review.** The **RREF** (row-reduced echelon form) of  $A$  is obtained from an echelon form by

- scaling the pivots to 1, and then
- eliminating the entries above the pivots.

A typical RREF has the shape

[\* represents an entry that could be anything]

$$\begin{bmatrix} 1 & * & 0 & * & * & 0 & * \\ & & 1 & * & * & 0 & * \\ & & & & & 1 & * \end{bmatrix}$$

**Example 8.** The RREF of  $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$  from the previous example is the  $2 \times 2$  identity matrix.

**Comment.** That's not surprising: A square matrix is invertible if and only if its RREF is the identity matrix. If that isn't obvious to you, think about how you invert a matrix using Gaussian elimination (reviewed next).

**Review.** Recall the Gauss–Jordan method of computing  $A^{-1}$ . Starting with the augmented matrix  $[A \mid I]$ , we do Gaussian elimination until we obtain the RREF, which will be of the form  $[I \mid A^{-1}]$  so that we can read off  $A^{-1}$ .

**Why does that work?** By our discussion, the steps of Gaussian elimination can be expressed by multiplication (on the left) with a matrix  $B$ . Only looking at the first part of the augmented matrix, and since the RREF of an invertible matrix is  $I$ , we have  $BA = I$ , which means that we must have  $B = A^{-1}$ . The other part of the augmented matrix (which is  $I$  initially) gets multiplied with  $B = A^{-1}$  as well, so that, in the end, it is  $BI = A^{-1}$ . That's why we can read off  $A^{-1}$ !

**For instance.** To invert  $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$  using the Gauss–Jordan method, we would proceed as follows

$$\left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & -6 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - 2R_1 \Rightarrow R_2} \left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -8 & -2 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} \frac{1}{2}R_1 \Rightarrow R_1 \\ -\frac{1}{8}R_2 \Rightarrow R_2 \end{array}} \left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array} \right] \xrightarrow{R_1 - \frac{1}{2}R_2 \Rightarrow R_1} \left[ \begin{array}{cc|cc} 1 & 0 & \frac{3}{8} & \frac{1}{16} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \end{array} \right]$$

and conclude that  $\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{16} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$ .

Of course, for  $2 \times 2$  matrices it is much simpler to use the formula  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

**Example 9. (extra)** Determine the LU decomposition of  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ .

**Solution.**  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1 \Rightarrow R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$  translates into  $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ .

Since  $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$  (no calculation needed!), we therefore have  $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ .