Example 70. One practical application of the QR decomposition is solving systems of linear equations.

$$A\boldsymbol{x}=\boldsymbol{b} \iff QR\boldsymbol{x}=\boldsymbol{b} \qquad \text{(now, multiply with } Q^T \text{ from the left)}$$
$$\implies R\boldsymbol{x}=Q^T\boldsymbol{b}$$

The last system is triangular and can be solved by back substitution.

A couple of comments are in order:

- If A is $n\times n$ and invertible, then the "$\implies$" is actually a "$\iff$".
- The equation $R\boldsymbol{x}=Q^T\boldsymbol{b}$ is always consistent! (Recall that R is invertible.)
  Indeed, if A is not $n\times n$ or not invertible, then $R\boldsymbol{x}=Q^T\boldsymbol{b}$ gives the least squares solutions!
  Why? $A^TA\hat{\boldsymbol{x}}=A^T\boldsymbol{b} \iff \underbrace{(QR)^TQR}_{=R^TQ^TQR}\hat{\boldsymbol{x}}=(QR)^T\boldsymbol{b} \iff R^TR\hat{\boldsymbol{x}}=R^TQ^T\boldsymbol{b} \iff R\hat{\boldsymbol{x}}=Q^T\boldsymbol{b}$
  [For the last step we need that R is invertible, which is always the case when A is $m\times n$ of rank n.]
- So, how does the QR way of solving linear systems compare to our beloved Gaussian elimination (LU)?
  It turns out that QR is a little slower than LU but makes up for it in "numerical stability".
  What does that mean? When computing numerically, we use floating point arithmetic and approximate each number by an expression of the form $0.1234\cdot 10^{-16}$. A certain (fixed) number of bits is used to store the part 0.1234 (here, 4 decimal places of accuracy) as well as the exponent -16.
  Now, here is something terrible that can happen in numerical computations: mathematically, the quantities x and $(x+1)-1$ are exactly the same. However, numerically, they might not. Take, for instance, $x=0.1234\cdot 10^{-6}$. Then, to an accuracy of 4 decimal places, $x+1=0.1000\cdot 10^{1}$, so that $(x+1)-1=0.0000$. But $x\neq 0$. We completely lost all the information about x.
  To be numerically stable, an algorithm must avoid issues like that.

$\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x}=\boldsymbol{b}$
$\iff R\hat{\boldsymbol{x}}=Q^T\boldsymbol{b}$ (where $A=QR$)

Example 71. Suppose Q has orthonormal columns. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(Q)$?

Solution. Recall that, to project onto $\mathrm{Col}(A)$, the projection matrix is $P=A(A^TA)^{-1}A^T$.
Since $Q^TQ=I$, to project onto $\mathrm{Col}(Q)$, the projection matrix is $P=QQ^T$.
Comment. A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q}\cdot\boldsymbol{b})\boldsymbol{q}=\boldsymbol{q}(\boldsymbol{q}^T\boldsymbol{b})=(\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.
Comment. In particular, if Q is not square, then $Q^TQ=I$ but $QQ^T\neq I$. In some sense, QQ^T still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\mathrm{col}(Q)$ it does act like the identity for vectors in $\mathrm{col}(Q)$. (Vectors not in $\mathrm{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\mathrm{col}(Q)$.)

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Example 72. Suppose A is invertible. What is the projection matrix P for orthogonally projecting onto $\mathrm{col}(A)$?

Solution. If A is an invertible $n \times n$ matrix, then $\mathrm{col}(A) = \mathbb{R}^n$ (because the n columns of A are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\mathrm{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

Example 73. What can we say about $\det(Q)$ if Q is orthogonal?

Solution. Write $d = \det(Q)$. Since $Q^{-1} = Q^T$, we have $\frac{1}{d} = d$ (recall that $\det(Q^{-1}) = 1/\det(Q)$ and $\det(Q^T) = \det(Q)$) or, equivalently, $d^2 = 1$. Hence, $d = \pm 1$.

Both of these are possible as the examples $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ illustrate.

Example 74. (homework) Diagonalize, if possible, the matrices

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and

(b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We will briefly discuss these next class.