Review. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthogonal, the orthogonal projection of $\boldsymbol{w}$ onto $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is

$$\hat{\boldsymbol{w}} = \frac{\boldsymbol{w}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\boldsymbol{v}_1 + \ldots + \frac{\boldsymbol{w}\cdot\boldsymbol{v}_n}{\boldsymbol{v}_n\cdot\boldsymbol{v}_n}\boldsymbol{v}_n.$$

Example 59. Determine the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Comment. We know how to do this using least squares.

However, realizing that $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal makes things even easier.

Solution. (using orthogonality) As in Example 58, the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is $-2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is $4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the orthogonal projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is $-2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$.

Important note. Note that, at this point, we can easily extend $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to an orthogonal basis of $\mathbb{R}^3$:

That is because the error $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is orthogonal to both of the existing basis vectors.

Therefore $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is an orthogonal basis of $\mathbb{R}^3$.

Gram–Schmidt

This idea (see "important note" above) for creating orthogonal vectors underlies Gram–Schmidt:

(Gram–Schmidt orthogonalization)

Given a basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ for W, produce an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ for W.

- $\boldsymbol{q}_1 = \boldsymbol{w}_1$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \left(\begin{array}{c}\text{projection of}\\ \boldsymbol{w}_2 \text{ onto } \boldsymbol{q}_1\end{array}\right)$
- $\boldsymbol{q}_3 = \boldsymbol{w}_3 - \left(\begin{array}{c}\text{projection of}\\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_1\end{array}\right) - \left(\begin{array}{c}\text{projection of}\\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_2\end{array}\right)$
- $\boldsymbol{q}_4 = \ldots$

Comment. Since $\boldsymbol{q}_1, \boldsymbol{q}_2$ are orthogonal, $\left(\begin{array}{c}\text{projection of}\\ \boldsymbol{w}_3 \text{ onto } \operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\}\end{array}\right) = \left(\begin{array}{c}\text{projection of}\\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_1\end{array}\right) + \left(\begin{array}{c}\text{projection of}\\ \boldsymbol{w}_3 \text{ onto } \boldsymbol{q}_2\end{array}\right)$.

Important comment. When working numerically it actually saves time to compute an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ by the same approach but always normalizing each $\boldsymbol{q}_i$ along the way. The reason this saves time is that now the projections onto $\boldsymbol{q}_i$ only require a single dot product (instead of two). This is called **Gram–Schmidt orthonormalization**.

Note. When normalizing, the orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ is the unique one with the property that $\operatorname{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_k\} = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ for all $k = 1, 2, \ldots$.

Armin Straub
straub@southalabama.edu

Example 60. Find an orthogonal basis for $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Solution. We already have the basis $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ for W. However, that basis is not orthogonal.

We can construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for W as follows:

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

  Since this is our first basis vector, we don't yet have other basis vectors it needs to be orthogonal to.

- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \begin{pmatrix} \text{projection of} \\ \boldsymbol{w}_2 \text{ onto } \boldsymbol{q}_1 \end{pmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 2/3 \end{bmatrix}$

  Make sure our way to construct $\boldsymbol{q}_2$ makes sense to you!

  $\boldsymbol{q}_2$ is the error of the projection of $\boldsymbol{w}_2$ onto $\boldsymbol{q}_1$. This guarantees that it is orthogonal to $\boldsymbol{q}_1$.

  On the other hand, since $\boldsymbol{q}_2$ is a combination of $\boldsymbol{w}_2$ and $\boldsymbol{q}_1$, we know that $\boldsymbol{q}_2$ actually is in W.

We have thus found the orthogonal basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -4/3 \\ 2/3 \end{bmatrix}$ for W.

Important comment. Normalizing these, we get $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, which is an orthonormal basis for W.

Comment. There are, of course, many orthogonal bases $\boldsymbol{q}_1, \boldsymbol{q}_2$ for W. Up to the length of the vectors, ours is the unique one with the property that $\text{span}\{\boldsymbol{q}_1\} = \text{span}\{\boldsymbol{w}_1\}$ and $\text{span}\{\boldsymbol{q}_1, \boldsymbol{q}_2\} = \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$.

Armin Straub
straub@southalabama.edu